STABILITY OF ABSTRACT BOUNDARY-VALUE PROBLEMS IN A BOUNDED DOMAIN UNDER WEAK PERTURBATIONS OF BOUNDARY OPERATORS\

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Abstract

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MATHEMATICS

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STABILITY OF ABSTRACT BOUNDARY-VALUE PROBLEMS IN A BOUNDED DOMAIN UNDER WEAK PERTURBATIONS OF BOUNDARY OPERATORS*

(Presented by Academician V. I. Smirnov on 13 V 1967)

Let

$$\mathcal{P}(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

be a differential expression with complex-valued coefficients, given in a bounded domain Ω of Euclidean space E_n, $\Gamma = \partial\Omega$**. Denote by P_0, P the minimal and maximal operators generated by $\mathcal{P}(x, D)$ in $L_2(\Omega)$, and let $\hat{P}$ ($P_0 \subset \hat{P} \subset P$; $D(\hat{P}) = \hat{D}$) be a closed extension of P_0. Put $\hat{\alpha} = \dim\ker \hat{P}$, $\hat{\beta} = \dim\operatorname{coker} \hat{P}$.

In the present paper, under the assumption that $\hat{P}$ is normally solvable and at least one of the dimensions $\hat{\alpha}, \hat{\beta}$ is finite, sufficient conditions are obtained for the stability of the property of normal solvability and of the index of the operator $\hat{P}$ under weak, in a certain sense, perturbations of its domain of definition. The article is adjacent to the works (^{5}), where analogous questions were treated for small perturbations.

1. Consider the Hilbert space H_P of elements $g(x) \in D(P)$ with scalar product $[g, h] = (g, h) + (Pg, Ph)$, where $(\,\cdot\,,\,\cdot\,)$ is the scalar product in $L_2(\Omega)$. Analogously we construct the space $H_{\bar{P}}$ for the differential expression $\overline{\mathcal{P}}(x, D)$ formally adjoint to $\mathcal{P}(x, D)$. The normal solvability of $\hat{P}$ is equivalent to the representation ((^{5}), Lemma 1)

$$H_P = \hat{D} \dotplus \hat{V},$$

$$\hat{V}\{v : v \in H_P,\ (\hat{P}\varphi, Pv) = 0,\ \forall\varphi \in \hat{D};\ (\varphi_0, v) = 0,\ \forall\varphi_0 \in \ker\hat{P}\}. \qquad (1)$$

Along with this, we shall consider the representation $H_{\bar{P}} = \hat{D}^* \dotplus \hat{V}^*$ ($\hat{D}^* = D(\hat{P}^*)$, $\hat{P}^*$ is the operator adjoint to $\hat{P}$ in $L_2(\Omega)$), which is constructed from the operator $\hat{P}^*$ in the same way as the representation (1) is constructed from the operator $\hat{P}$. Denote by $\hat{\chi}$ ($\hat{\chi}^*$) the projection operator of H_P ($H_{\bar{P}}$) onto $\hat{V}$ ($\hat{V}^*$) parallel to $\hat{D}$ ($\hat{D}^*$).

Let $\widetilde{D}$ ($D(P_0) \subset \widetilde{D} \subset H_P$) be another subspace and let $\widetilde{P}$ be the restriction of P to $\widetilde{D}$; $\widetilde{D}^* = D(\widetilde{P}^*)$, $\widetilde{P}^*$ is the operator adjoint in $L_2(\Omega)$ to $\widetilde{P}$; $\tilde{\alpha} = \dim \ker \widetilde{P}$, $\tilde{\beta} = \dim \operatorname{coker} \widetilde{P}$.

Theorem 1. *Let $\hat{P}$ be normally solvable. 1) If $\hat{\alpha} < \infty$ and the operator $\hat{\chi} : \widetilde{D} \to L_2(\Omega)$ is completely continuous, then $\widetilde{P}$ is normally solvable and $\tilde{\alpha} < \infty$. 2) If $\hat{\beta} < \infty$ and the operator $\hat{\chi}^* : \widetilde{D}^* \to L_2(\Omega)$ is completely continuous, then $\widetilde{P}$ is normally solvable and $\tilde{\beta} < \infty$.*

For arbitrary $\Phi_{\pm}$-operators in a Banach space, a similar result was obtained by G. Neubauer ((1), Theorem 6.7). From this result the conclusions of Theorem 1 follow only under the assumption of complete continuity of the operators $\hat{\chi} : \widetilde{D} \to H_P$, $\hat{\chi}^* : \widetilde{D}^* \to H_{\bar{P}}$.

* Work on this article is connected with a question posed to the author by M. Sh. Birman.
** The coefficients $a_\alpha(x)$ and the boundary Γ are assumed to be sufficiently smooth.

From Theorem 1 there follows the possibility of constructing a decomposition $H_P = \hat{D} + \hat{V}$ of type (1) for the operator $\hat{P}$. Let $\varkappa$ be the projection operator of H_P onto $\hat{V}$ parallel to $\hat{D}$.

Definition 1. We shall call the subspaces $\hat{D}$ and $\tilde{D}$ *completely continuous perturbations* of one another if the operators $\hat{\varkappa} : \tilde{D} \to H_P$ and $\tilde{\varkappa} : \hat{D} \to H_P$ are completely continuous.

Apparently, for the first time (and in a more general situation) this class of perturbations was studied in (1). That, with respect to perturbations of this type, $\operatorname{ind} \hat{P}$ is not stable is shown by the following

Example. Let $N = \ker P$, $\hat{D}$ be given, $\hat{N} = \ker \hat{P}$, and $\tilde{D} = \hat{D} \oplus \{\lambda\psi\}$, where $\{\lambda\psi\}$ is a one-dimensional subspace generated by an arbitrary element $\psi \in N \ominus \hat{N}$. Then $\hat{\varkappa}\tilde{D} = \{\lambda\psi\}$, $\tilde{\varkappa}\hat{D} = \{0\}$; nevertheless $\operatorname{ind} \tilde{P} = \operatorname{ind} \hat{P} + 1$.

Denote by $\hat{V}^\perp, \tilde{V}^\perp$ the $[\cdot,\cdot]$-orthogonal complements, respectively, of $\hat{D}, \tilde{D}$ in H_P.

Theorem 2. *Let $\hat{P}$ be a Φ ($\Phi_\pm$)-operator and let the subspaces $\hat{D}, \tilde{D}$ be completely continuous perturbations of one another. In order that* $\operatorname{ind} \tilde{P} = \operatorname{ind} \hat{P}$, *it is necessary and sufficient that*

$$\dim \tilde{D} \cap \hat{V}^\perp = \dim \hat{D} \cap \tilde{V}^\perp. \tag{2}$$

It is verified directly that

$$\tilde{D} \cap \hat{V}^{\perp} = \ker(\tilde{P}^{*}\hat{P} + I), \qquad \tilde{D} \cap \tilde{V}^{\perp} = \ker(\hat{P}^{*}\tilde{P} + I).$$

We note that, for perturbations small in the sense of (5), both dimensions in formula (2) are equal to zero.

2. Denote by $\hat{\varkappa}^{\perp}, \tilde{\varkappa}^{\perp}$ the $[\cdot,\cdot]$-orthoprojectors, respectively, onto $\hat{V}^{\perp}$ and $\tilde{V}^{\perp}$. It can be shown that the subspaces $\hat{D}, \tilde{D}$ are completely continuous perturbations of one another if and only if the operator $(\hat{\varkappa}^{\perp} - \tilde{\varkappa}^{\perp}) : H_P \to H_P$ is completely continuous. The latter condition is equivalent to (2, Lemma 3.14) the simultaneous complete continuity of the operators $R[\hat{P}] - R[\tilde{P}]$, $R[\hat{P}^{*}] - R[\tilde{P}^{*}]$, and $\hat{P}R[\hat{P}] - \tilde{P}R[\tilde{P}]$ in $L_2(\Omega)$, where, by definition,

$$R[K] = (K^{*}K + I)^{-1}$$

for any densely defined and closed operator K in $L_2(\Omega)$ (below K will denote an operator with these properties).

Consider in $L_2(\Omega)$ the positive self-adjoint operator

$$S[K] = (R[K])^{1/2}.$$

Lemma 1 (Cordes and Labrousse (2)). *1) The operators K and $KS[K]$ are simultaneously normally solvable or not normally solvable. 2)* $\ker K = \ker KS[K]$*;* $\operatorname{coker} K = \operatorname{coker} KS[K]$.

Definition 2. We shall call the subspaces D and $\tilde{D}$ *weak perturbations* of one another if the operator

$$(\hat{P}S[\hat{P}] - \tilde{P}S[\tilde{P}]) : L_2(\Omega) \to L_2(\Omega)$$

is completely continuous.

Theorem 3. *Let $\hat{P}$ be a Φ ($\Phi_{\pm}$)-operator and let the subspaces $\hat{D}$ and $\tilde{D}$ be weak perturbations of one another. Then $\tilde{P}$ is also a Φ ($\Phi_{\pm}$)-operator and* $\operatorname{ind} \tilde{P} = \operatorname{ind} \hat{P}$.

This theorem follows directly from Lemma 1 and the known assertions of perturbation theory for Φ ($\Phi_{\pm}$)-operators by completely continuous operators (3,4).

Corollary. *The conclusions of Theorem 3 are valid if the operator*

$$(S[\hat{P}] - S[\tilde{P}]) : L_2(\Omega) \to H_P$$

is completely continuous.

3. As in the papers (5), introduce boundary operators $\hat{A}, \hat{B}, \dots, \hat{A}'$, proceeding from the representations

$$(Pu, w) - (u, \tilde{P}w) = \langle \hat{A}u|_{\Gamma}, \hat{B}w|_{\Gamma} \rangle + \langle \hat{B}u|_{\Gamma}, \hat{A}'w|_{\Gamma} \rangle = \tag{3}$$

$$= \langle \tilde{A}u|_{\Gamma}, \tilde{B}w|_{\Gamma} \rangle + \langle \tilde{B}'u|_{\Gamma}, \tilde{A}'w|_{\Gamma} \rangle,$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in some Hilbert space $H(\Gamma)$ of vector functions defined on Γ, such that the operators $\hat P, \tilde P$ ($\hat P^*, \tilde P^*$) are the restrictions of P ($\bar P$), respectively, to the subspaces

$$\hat D = \{\varphi : \hat A\varphi|_\Gamma = 0\}, \quad \tilde D = \{\psi : \tilde A\psi|_\Gamma = 0\}$$

$$(\hat D^* = \{\varphi^* : \hat A'\varphi^*|_\Gamma = 0\}, \quad \tilde D^* = \{\psi^* : \tilde A'\psi|_\Gamma = 0\}).$$

Put

$$\hat Q = (S[\hat P])^{-1}, \quad \tilde Q = (S[\tilde P])^{-1}, \quad T = \hat Q - \tilde Q, \quad Q = \hat Q + \tilde Q.$$

Then from the representations (3) we obtain

$$\left|(S[\hat P] - S[\tilde P])f\right|^2 = (Tf, (S[\hat P] - S[\tilde P])f) + \gamma^*; \tag{4}$$

$$\begin{aligned}\gamma = {} & \langle \tilde B' P(S[\hat P] - S[\tilde P])f|_\Gamma, \tilde A(S[\hat P] - S[\tilde P])f|_\Gamma\rangle \\ & + \langle \tilde B'(S[\hat P] - S[\tilde P])f|_\Gamma, \tilde A' P(S[\hat P] - S[\tilde P])f|_\Gamma\rangle.\end{aligned} \tag{5}$$

We shall assume that the operators occurring in formulas (4), (5) have the following properties: 1) the restrictions of the operators T and Q to $C_0^\infty(\Omega)$ are pseudodifferential operators**; 2) the symbols of these operators $t(x,\xi)$ and $q(x,\xi)$ admit asymptotic expansions with homogeneous in ξ terms $t_j(x,\xi)$ and $q_j(x,\xi)$ of orders, respectively, $\tau_j \searrow -\infty$ and $\chi_j \searrow -\infty$, such that the function $h(j,k) = \chi_j + \tau_k$ satisfies the condition: if $h(j_0,k_0) \geq 0$ and $j_1 + k_1 > j_0 + k_0$, then $h(j_1,k_1) < h(j_0,k_0)$; 3) if the sequences $\{\varphi_n\} \subset \hat D$ and $\{\psi_n\} \subset \tilde D$ are such that $\{\varphi_n - \psi_n\}$ is bounded in H_P, then $\{\tilde B'(\varphi_n - \psi_n)|_\Gamma\}$ is a bounded sequence and $\{\tilde A\varphi_n|_\Gamma\}$ is a compact sequence in $H(\Gamma)$; 4) if the sequences $\{\varphi_n^*\} \subset \hat D^*$ and $\{\psi_n^*\} \subset \tilde D^*$ are such that $\{\varphi_n^* - \psi_n^*\}$ is bounded in H_P, then $\{\tilde B'(\varphi_n^* - \psi_n^*)|_\Gamma\}$ is a bounded sequence and $\{\tilde A'\varphi_n^*|_\Gamma\}$ is a compact sequence in $H(\Gamma)$.

Lemma 2. *Let conditions 1), 2) be fulfilled. Then the operator*

$$T : C_0^\infty(\Omega) \cap L_2(\Omega) \to L_2(\Omega)$$

is completely continuous.

Remark. If $j_1 + k_1 > j_0 + k_0$ implies $h(j_1,k_1) < h(j_0,k_0)$ for all integers j_0, k_0, then $T \in \mathcal{L}_{-\infty}$ (i.e. $t_j(x,\xi) \equiv 0,\ j = 0,1,2,\ldots$).

Lemma 2 and the remark to it are derived from the obvious relation $QT = -TQ$ by constructing an asymptotic series for the symbol of the operator QT.

Theorem 4. *Let the conditions 1)–4) formulated above be fulfilled. Then the subspaces $\hat D$ and $\tilde D$ are weak perturbations of one another, the operators $\hat P$ and $\tilde P$ either both are not, or both are, Φ-($\Phi_\pm$)-operators, and in the latter case*

$$\operatorname{ind} \tilde P = \operatorname{ind} \hat P.$$

The proof of this theorem is based on the corollary to Theorem 3, Lemma 2, and an analysis of formulas (4), (5).

Let us note that the conclusions of Lemma 2 and Theorem 4 remain valid if conditions 1), 2) are replaced by the following: on $C_0^\infty(\Omega)$ the operators $\hat{Q}, \tilde{Q}$ and Q are elliptic pseudodifferential operators of order m, while $S[\hat{P}]$, $S[\tilde{P}]$, and Q^{-1} are of order $-m$.

In the case when the coefficients $a_\alpha(x)$ are constant, condition 1) is fulfilled, and $T \in \mathcal{L}_{-\infty}$. Thus, in this case the conclusions of Theorem 4 follow from only the conditions 3), 4) on the boundary operators.

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* $|\cdot|$ is the norm of H_P.
** In the sense of (6, 7).