GROUPS SATISFYING THE WEAK MINIMAL CONDITION

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GROUPS SATISFYING THE WEAK MINIMAL CONDITION

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By a finiteness condition in group theory is meant a property, possessed by all finite groups, such that there exists at least one infinite group that does not possess this property (see ([1])).

The systematic study of groups with various finiteness conditions began about 30 years ago and was to a considerable extent connected with the investigation of locally soluble groups satisfying the minimal condition for subgroups (see ([2–5])). The minimal condition for subgroups played a significant role in the formation of the modern direction in group theory connected with generalized nilpotent and generalized soluble groups (see ([1])).

However, the minimal condition is a very strong restriction, since none of the nonperiodic groups, obviously, satisfies it. In the present paper an attempt is made to weaken it somewhat in this respect.

Definition 1. We shall say that a group $\mathfrak{G}$ satisfies the weak minimal condition for subgroups if in it there does not exist an infinite descending chain of subgroups

$$\mathfrak{G}_1 \supset \mathfrak{G}_2 \supset \cdots \supset \mathfrak{G}_k \supset \mathfrak{G}_{k+1} \supset \cdots,$$

satisfying the following condition: the index $[\mathfrak{G}_k : \mathfrak{G}_{k+1}]$ of the subgroup $\mathfrak{G}_{k+1}$ in the group $\mathfrak{G}_k$ is infinite ($k = 1, 2, \ldots$).

A group $\mathfrak{G}$ satisfying the weak minimal condition for subgroups need not be periodic, since, for example, an infinite cyclic group satisfies the weak minimal condition. However, in the case of locally finite groups the weak minimal condition is equivalent to the ordinary minimal condition for subgroups (Theorem 1).

1. It is clear that every subgroup and factor group of a group satisfying the weak minimal condition for subgroups also satisfies the weak minimal condition for subgroups. It turns out, further, that the class of groups

satisfying the weak minimal condition is closed also with respect to the operation of extension.

Lemma 1. If a normal divisor $\mathfrak{N}$ and the factor group $\mathfrak{G}/\mathfrak{N}$ satisfy the weak minimal condition, then the group $\mathfrak{G}$ also satisfies the weak minimal condition for subgroups.

The following auxiliary assertion holds:

Lemma 2. If a locally finite group $\mathfrak{G}$ has an infinite descending sequence of subgroups, then the group $\mathfrak{G}$ has an infinite subgroup of infinite index.

With the aid of Lemmas 1 and 2 one obtains

Theorem 1. For locally finite groups the minimal condition and the weak minimal condition for subgroups are equivalent.

2. We pass to the study of the structure of locally soluble groups satisfying the weak minimal condition.

Lemma 3. If a group $\mathfrak{G}$ has an infinite descending sequence of characteristic subgroups of finite index with trivial

intersection, then the automorphism group $\mathfrak{F}$ of the group $\mathfrak{G}$ has a descending chain of normal divisors of finite index with trivial intersection.

Lemma 4. If an infinite quasicyclic (i.e. having no proper subgroups of finite index) locally soluble group $\mathfrak{G}$ satisfies the weak minimal condition for subgroups, then $\mathfrak{G}$ is a direct product of a finite number of quasicyclic groups*.

Using these propositions, one can establish the following theorem, which is very helpful in the study of locally soluble groups with the weak minimal condition for subgroups.

Theorem 2. A locally soluble group $\mathfrak{G}$ satisfying the weak minimal condition for subgroups is soluble.

Definition 2. Following the book of A. G. Kurosh ([6]), we shall call a **characteristic** any sequence of the form

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n, \dots),$$

where each α_n is either zero, or some natural number, or the symbol ∞. On the set of characteristics an equivalence relation is introduced (see ([6])). A class of equivalent characteristics is called a **type**. A type determined by a characteristic of the form

$$(\alpha_1, \alpha_2, \dots, \alpha_n, 0, 0, \dots),$$

will be called a **finite type**.

In [6] a one-to-one correspondence is established between the set of all types and the set of all nonisomorphic torsion-free groups of rank 1. A torsion-free group corresponding in this way to a finite type will be called a **rational group of finite type**. It can be shown that rational groups of finite type, and only they, are torsion-free groups of rank 1 satisfying the weak minimal condition for subgroups.

The following theorem clarifies the structure of locally soluble groups with the weak minimal condition for subgroups.

Theorem 3. A locally soluble group $\mathfrak{G}$ satisfies the weak minimal condition for subgroups if and only if it contains a normal divisor of finite index $\mathfrak{H}$ such that:

1) all elements of finite order of the group $\mathfrak{H}$ form a subgroup $\mathfrak{R}$, decomposable into a direct product of a finite number of quasicyclic groups;
2) the factor group $\mathfrak{H}/\mathfrak{R}$ has a finite rational series (see [1]), whose factors are rational groups of finite type.

Remark 1. One can prove even more, namely: the factor group $\mathfrak{H}/\mathfrak{R}$ has such a finite series of subgroups, invariant in $\mathfrak{G}/\mathfrak{R}$, whose factors are abelian torsion-free groups satisfying the weak minimal condition for subgroups.

2. In particular, if $\mathfrak{G}$ is a periodic group, Theorem 3 turns into the known description of locally soluble groups satisfying the minimal condition for subgroups (see [3]).

Let us give one more characterization of the class of locally soluble groups satisfying the weak minimal condition for subgroups.

Theorem 3*. The class of locally soluble groups with the weak minimal condition for subgroups is the smallest class of groups closed under the operation of extension and containing all locally soluble

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* Let us note that in the proof of the lemma only the following property of an infinite locally soluble group was used: every infinite factor group of an arbitrary infinite subgroup has a proper normal divisor. Groups with this property are called H-groups in [1].

groups every proper subgroup of which has a finite number of generators.

Corollary. A locally soluble group all of whose proper subgroups have a finite number of generators either satisfies the maximal condition for subgroups or is a quasicyclic group.

This theorem indicates one more analogy between the class of locally soluble groups with the minimal condition for subgroups and the class of locally soluble groups with the weak minimal condition for subgroups. Indeed, as can be shown, the class of locally soluble groups with the minimal condition for subgroups is the smallest class of groups, closed with respect to the operation of extension and containing all locally soluble groups every proper subgroup of which is finite.

3. It is known (see (1)) that a locally soluble group satisfying the minimal condition for abelian subgroups also satisfies the minimal condition for all subgroups. In this section an analogous result is obtained for the case of radical groups satisfying the weak minimal condition for abelian subgroups.

Theorem 4. If a locally nilpotent torsion-free group $\mathfrak{G}$ satisfies the weak minimal condition on abelian subgroups, then $\mathfrak{G}$ is a nilpotent group satisfying the minimal condition on all subgroups and has a finite invariant series whose factors are rational groups of finite type.

This theorem is in a certain sense analogous to the well-known theorem of A. I. Mal' cev (9): *if all abelian subgroups of a locally nilpotent torsion-free group have finite ranks, then the group itself is a nilpotent group of finite rank.* In contrast to the latter, Theorem 4 is proved by purely group-theoretic methods. In its proof the following auxiliary proposition was used:

Lemma 5. Let $\mathfrak{A}, \mathfrak{B}$ be two abelian torsion-free groups satisfying the weak minimal condition for subgroups. Then the group of homomorphisms of the group $\mathfrak{A}$ into the group $\mathfrak{B}$ also satisfies the weak minimal condition for subgroups.

Theorem 5. Let $\mathfrak{G}$ be a radical group (in the sense of B. I. Plotkin), and let all its abelian subgroups satisfy the weak minimal condition for subgroups. Then the group $\mathfrak{G}$ satisfies the weak minimal condition for all subgroups.

Since every radical group is locally soluble, the structure of a group $\mathfrak{G}$ satisfying the condition of this theorem is determined by Theorem 3.

We note that an abelian group satisfying the weak minimal condition is an abelian group of finite rank; therefore Theorem 5 is directly related to the problem of studying groups all of whose abelian subgroups have finite rank (see (7)).

The proof of the theorem uses results from the works of V. S. Charin (8), M. I. Kargapolov (9), and B. I. Plotkin (10).

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Note: Figure translations are in progress. See original paper for figures.