ON A LINEAR FORM IN LOGARITHMS OF ALGEBRAIC NUMBERS

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Abstract

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By the **length** $L(\xi)$ of an algebraic number ξ we shall mean the sum of the moduli of the coefficients of the equation defining the number ξ.

Let K be the field obtained by adjoining the algebraic numbers $\alpha_1, \dots, \alpha_m$; $\beta_0, \beta_1, \dots, \beta_m$ to the field of rational numbers, and let n be its degree,

$$h = \max\{L(\alpha_1), \dots, L(\alpha_m), e^{|\ln \alpha_1|}, \dots, e^{|\ln \alpha_m|}\},$$

$$H = \max L(\beta_i), \qquad i = 0, 1, \dots, m.$$

There exists an effective absolute constant c_0 such that, if the quantities $\ln \alpha_1, \dots, \ln \alpha_m$ are linearly independent over the field of rational numbers, and if $|\beta_0| + |\beta_1| + \dots + |\beta_m| > 0$, then the inequality

$$|\beta_0 + \beta_1 \ln \alpha_1 + \dots + \beta_m \ln \alpha_m| > \exp\left[-(c_0 + 90^{m^2} n \ln h)^{16m^2 \ln H}\right]. \quad (1)$$

holds.

Interest in effective estimates of type (1) arose long ago, since, besides their independent significance, they make it possible to effectivize the solution of a number of problems in number theory (see, for example, ([1])).

For $m = 2$, for a homogeneous form a similar estimate was first obtained in 1935 by A. O. Gelfond. For its derivation he used his interpolation method, created by him for the solution of the well-known seventh Hilbert problem. Successively strengthening his estimate, A. O. Gelfond obtained ([2]) the inequality

$$|\beta_1 \ln \alpha_1 + \beta_2 \ln \alpha_2| > \exp[-n^2 \ln^{2+\varepsilon}(n + \ln H)], \quad (2)$$

$$\varepsilon > 0, \qquad n + \ln H \geq H_0(\ln \alpha_1, \ln \alpha_2, \varepsilon).$$

For an arbitrary natural m, A. O. Gelfond obtained ([3]) the inequality

$$|x_1 \ln \alpha_1 + \dots + x_m \ln \alpha_m| > e^{-\varepsilon x},$$

$$\varepsilon > 0, \qquad x = \max_{1 \le i \le m} |x_i| \ge x_0(\ln \alpha_1, \dots, \ln \alpha_m, \varepsilon).$$

Here $x_1, \dots, x_m$ are rational integers. This inequality was obtained by the Thue-Siegel method, so that the constant x_0 is ineffective.

The first effective estimates of the form $\beta_1 \ln \alpha_1 + \dots + \beta_m \ln \alpha_m$ were obtained by A. Baker ([4–6]). In ([4]) he obtained the inequality

$$|\beta_1 \ln \alpha_1 + \dots + \beta_m \ln \alpha_m| > C \exp[-\ln^{\chi} H], \tag{3}$$

where $\chi > m + 1$, the constant C depends on $\chi, n, \ln \alpha_1, \dots, \ln \alpha_m$, and the numbers $\ln \alpha_1, \dots, \ln \alpha_m$ and πi are linearly independent over the field of rational numbers.

In ([5]) linear independence is required only for the numbers $\ln \alpha_1, \dots, \ln \alpha_m$ themselves, but here $\chi > 2m + 1$. In ([6]) A. Baker extended his results to nonhomogeneous forms.

A. Baker proved his theorems by means of the above-mentioned method of A. O. Gelfond, adding to it an important additional consideration concerning the possibility of "many-fold" interpolation of the auxiliary form.

For the derivation of inequality (1), use is made primarily of A. O. Gelfond' s method, as well as of A. Baker' s additions and some further considerations.

In deriving inequality (1), the author sought to obtain the best estimate with respect to H. This "preference" for H corresponds to the case of fixed $\ln \alpha_1, \dots, \ln \alpha_m$ and bounded n (it is precisely this case that was considered in the works of A. O. Gelfond and A. Baker), or to the case in which h and n vary sufficiently slowly relative to H. In the case where all the quantities $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m$ vary independently, one can improve the estimate with respect to h and n by worsening it with respect to H.

An inequality analogous to inequality (1) is also valid for the p-adic metric.

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Note: Figure translations are in progress. See original paper for figures.