ON A. N. TIKHONOV' S FIXED-POINT PRINCIPLE IN EQUATIONS WITH OPERATORS WEAKLY CLOSED ON THE KERNEL

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R. I. KACHUROVSKII

ON A. N. TIKHONOV' S FIXED-POINT PRINCIPLE IN EQUATIONS WITH OPERATORS WEAKLY CLOSED ON THE KERNEL

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The well-known fixed-point principle of A. N. Tikhonov (^{1}) asserts that the equation $x - T(x) = \theta$ is solvable in a locally convex space under certain restrictions on the operator $T(x)$. Theorem 1 of the present paper gives other conditions for the solvability of the equation $F(x) = \theta$ with a nonlinear operator $F(x)$ acting from a locally convex space into its conjugate. This theorem may be regarded as a certain supplement to the principle mentioned. Theorem 2 gives conditions for the solvability of equations with operators that are weakly discontinuous in the whole space. In the second section the solvability of equations with operators weakly closed on the kernel is proved. The result obtained contains, as special cases, a number of known existence theorems, for example the theorems of M. A. Krasnosel' skii, F. E. Browder, and others. Finally, a theorem on the solvability of equations with pseudomonotone operators is obtained in the paper.

1. Let X be a locally convex linear topological space over the field of real numbers; X^* the set of all continuous linear functionals defined on X. Let S be a subset of X; $K(S)$ the closed convex hull of the set S. Let $\theta_X \in K(S) - S$, and for every finite-dimensional subspace $X^n \subset X$ let the boundary of the set $K(S) \cap X^n$ be contained in $S \cap X^n$. In this case we shall say that S surrounds θ_X. In what follows, the notation (z, x), for $z \in X^*, x \in X$, denotes the value of the linear functional z at the element x. The following lemma is known (^{6}).

Lemma 1. Let X be a finite-dimensional Euclidean space; Ω a bounded closed convex set in X containing zero as an interior point. Let the operator $T : \Omega \to X$ be continuous on Ω and satisfy the condition $(T(x), x) \geqslant 0$ on the boundary of Ω. Then there exists an element $u_0 \in \Omega$ such that $T(u_0) = 0$.

Theorem 1. Let X be a locally convex space, the set $S \subset X$ surround zero θ, and let its closed convex hull $K(S)$ be a bicompact set in X. Let the operator $F : K(S) \to X^*$ be continuous in the following sense: for every $u \in X$ the functional $\varphi_u(x) = (F(x), u)$ is continuous on the set $K(S)$ with respect to x.

Suppose the condition

$$(F(x), x) \geqslant 0, \quad \forall x \in S$$

is fulfilled. Then the equation $F(x) = \theta_{X^*}$ has at least one solution $x_0 \in K(S)$.

Proof. For fixed $u \in X$, consider the set

$$M_u = \{x : (F(x), u) = 0, \quad x \in K(S)\}.$$

By virtue of the continuity of F and the condition $(F(x), x) \geqslant 0$ on S, each set M_u is nonempty and closed, with $M_u \subset K(S)$. Let us show that M_u is nonempty. Consider the line $X^1 = \{tu,\ t \in R^1\}$ and the set $K(S) \cap X^1$. Since S surrounds θ, the point θ belongs to the interior of the set $K(S) \cap X^1$. Therefore on the boundary of this set there are elements $t'u$, $t''u$, where $t' < 0$, $t'' > 0$. Since the boundary of the set $K(S) \cap X^1$ is contained in

in $S \cap X^1$, then $(F(t'u), t'u) \geq 0$, $(F(t''u), t''u) \geq 0$ or $(F(t'u), u) \leq 0$, $(F(t''u), u) \geq 0$. By the continuity of F it follows from this that, for some t_0, $(t_0 u) \in K(S)$, the equality $(F(t_0 u), u) = 0$ holds, and this means that $(t_0 u) \in M_u$.

We shall show that the sets M_u, $u \in X$, form a centered family. To this end, fix $u_1, \dots, u_n \in X$ and consider the system of n equations in the n real unknowns $\alpha_1, \dots, \alpha_n$

$$\psi_j(\alpha) = \psi_j(\alpha_1, \dots, \alpha_n) = \left(F\left(\sum_{i=1}^{n} \alpha_i u_i\right), u_j\right) = 0, \qquad j = 1, \dots, n. \tag{1}$$

Without loss of generality, one may assume that $u_1, \dots, u_n$ are linearly independent. The totality of vectors $(\alpha_1, \dots, \alpha_n)$ such that the corresponding elements $\sum_{i=1}^{n} \alpha_i u_i$ fill the set $K(S) \cap X^n$ (where X^n is the linear span of the vectors $u_1, \dots, u_n$) is a bounded closed convex set Ω^n in n-dimensional Euclidean space. The boundary of the set Ω^n consists of all vectors $(\alpha_1, \dots, \alpha_n)$ for which $\sum_{i=1}^{n} \alpha_i u_i$ belongs to the boundary of the set $K(S) \cap X^n$, and θ is an interior point of the set Ω^n. Consider the operator $\psi(\alpha) = \{\psi_1(\alpha), \dots, \psi_n(\alpha)\}$, acting from Ω^n into n-dimensional Euclidean space. We have

$$(\psi(\alpha), \alpha) = \sum_{j=1}^{n} \psi_j(\alpha)\alpha_j = \sum_{j=1}^{n} \left(F\left(\sum_{i=1}^{n} \alpha_i u_i\right), u_j\right)\alpha_j = \left(F\left(\sum_{i=1}^{n} \alpha_i u_i\right), \sum_{i=1}^{n} \alpha_i u_i\right). \tag{2}$$

If the vector $\alpha = (\alpha_1, \dots, \alpha_n)$ lies on the boundary of the set Ω^n, then $\sum_{i=1}^{n} \alpha_i u_i$ lies on the boundary of the set $K(S) \cap X^n$, and therefore $\sum_{i}^{n} \alpha_i u_i \in S \cap X^n$. Since, by the hypothesis of the theorem, $(F(x), x) \geq 0$, $\forall x \in S$, on the boundary of the set Ω^n, according to equality (2), the inequality $(\psi(\alpha), \alpha) \geq 0$ holds. The continuity of the operator $\psi(\alpha)$ on Ω^n is obvious. By Lemma 1 there exists $\alpha^0 = (\alpha_1^0, \dots, \alpha_n^0) \in \Omega^n$ such that $\psi(\alpha^0) = \theta$, and moreover

$$u_0 = \sum_{i=1}^{n} \alpha_i^0 u_i \in K(S).$$

Since $\psi(\alpha^0) = \theta$, we have $\psi_j(\alpha^0) = 0$, whence it follows that $(F(u_0), u_j) = 0$ for all $j = 1, \dots, n$. Therefore $u_0 \in \bigcap_{j=1}^{n} M_{u_j}$. Thus the sets M_u form a centered family. Since $M_u \subset K(S)$ and $K(S)$ is a bicompact set, $\bigcap_{u \in X} M_u$ is nonempty. Let $x_0 \in \bigcap_{u \in X} M_u$. Then $x_0 \in K(S)$, $(F(x_0), u) = 0$ for all $u \in X$. Hence it follows that $F(x_0) = \theta_{X^*}$.

Theorem 2. *Let E be a reflexive real Banach space; let D be a bounded closed convex set in E, containing zero as an interior point. Let $F : D \to E^*$ be a weakly continuous operator (i.e., for every $y \in E$ the functional $\varphi_y(x) = (F(x), y)$ is continuous as an operator from the set D, considered with the weak topology, into R^1). Suppose that on the boundary of the set D the inequality $(F(x), x) \geq 0$ holds. Then the equation $F(x) = \theta_{E^*}$ has at least one solution in D.*

Proof. The space E, considered with the weak topology, is a locally convex space. Since D is a convex closed set, it is weakly closed, and since D is also bounded and the space E is reflexive, the set D is bicompact.

in the weak topology. To complete the proof it remains to apply Theorem 1.

In (^{3,4}) Theorem 2 is formulated for separable Banach spaces. Let us note that Theorem 2 admits the following rephrasing. Let E be a real reflexive Banach space, $F : E \to E^*$ a linear or nonlinear mapping, weakly continuous in every ball of E, satisfying the coercivity condition

$$(F(x), x) \geqslant \gamma(\|x\|)\|x\|, \quad \forall x \in E,$$

where $\gamma(t)$ is a real function of a nonnegative argument t, $\lim\limits_{t \to +\infty} \gamma(t) = +\infty$. Then the equation $F(x) = z$ has, for every $z \in E^*$, at least one solution $x_0 \in E$.

2. Everywhere in this section E is a separable reflexive Banach space over the field of real numbers; D is a subset of the space E; F is, in general, a nonlinear operator whose domain is the set D, and whose values are elements of the conjugate space E^*. We shall call the set

$$M = \{x : (F(x), x) = 0,\ x \in D\}$$

the kernel of the operator F. We shall call the operator F weakly closed on its kernel if from the relations

$$x_n \in M, \quad x_n \xrightarrow[\text{weakly}]{} x_0, \quad F(x_n) \xrightarrow[\text{weakly}]{} \theta$$

it follows that $x_0 \in M$, $F(x_0) = \theta$. The notation $F(x_n) \xrightarrow[\text{weakly}]{} \theta$ is to be understood as follows: for every $y \in E$,

$$\lim_{n\to\infty} (F(x_n), y) = 0.$$

These definitions, convenient in the situation considered here, are not generally accepted. Further, F is called bounded on the kernel (on the whole space E) if it maps every bounded subset of M (of E) into a bounded subset of E^*. The operator F is called finitely continuous on D if, for every natural n and any fixed $x_1, \dots, x_n \in E$, each function

$$\psi_j(a_1, \dots, a_n) = \left(F\left(\sum_{i=1}^{n} a_i x_i \right), x_j \right) \qquad (j = 1, \dots, n)$$

of n real variables is continuous at every point $(a_1, \dots, a_n)$ for which

$$\sum_{i=1}^{n} a_i x_i \in D.$$

Theorem 3. Let E be a real separable reflexive Banach space; let D be a bounded closed convex set in E, containing zero as an interior point. Let the operator $F : D \to E^*$ be finitely continuous in D, bounded and weakly closed on its kernel, and satisfy on the boundary of the set D the condition

$$(F(x), x) \geqslant 0.$$

Then the equation $F(x) = \theta$ has a solution $x_0 \in D$.

The following lemma gives conditions for weak closedness of an operator on its kernel.

Lemma 2. a) Let D be a weakly closed set in a separable Hilbert space H; $B : D \to H$ a completely continuous operator (i.e., continuous and compact). Then $F = I - B$ is weakly closed on its kernel ($Ix \equiv x$).

b) Let D and H be the same as in part a), $B : D \to H$ nonexpansive, i.e.

$$\|B(x) - B(y)\| \leqslant \|x - y\|, \quad \forall x, y \in D.$$

Then $F = I - B$ is weakly closed on its kernel.

c) Let D be a weakly closed set in E, $F : D \to E^*$ a weakly continuous operator. Then F is weakly closed on its kernel.

d) Let $F : E \to E^*$ be a ray-continuous monotone operator, i.e.

$$(F(x) - F(y), x - y) \geqslant 0, \quad \forall x, y \in E.$$

Then F is weakly closed on its kernel.

With the aid of Lemma 2, from Theorem 3 one can obtain a number of interesting corollaries. For example, for separable Hilbert spaces we obtain the contraction mapping and Schauder principles (for the sphere), F. E. Browder' s theorem on fixed points of nonexpansive mappings (^{5}), M. A. Krasnosel' skii' s theorem (^{2}), p. 314, on fixed points of completely continuous operators, a theorem similar to it with complete continuity replaced by weak continuity, and theorems on the solvability of equations with monotone and weakly continuous operators (^{3,4}).

We shall formulate one new result following from Theorem 3. We shall say that an operator $\Phi : E \to E^*$ satisfies the condition of pseudomonotonicity if

$$(\Phi(x_2) - \Phi(x_1), x_2 - x_1) + |(\Phi(x_1), x_2 - x_1)| \geq 0,$$

$\forall x_2, x_1 \in E$.

Theorem 4. Let E be a real separable reflexive Banach space; let the operator $F : E \to E^*$ be finitely continuous on all of E, bounded, and satisfy the coercivity condition

$$(F(x), x) \geq \gamma(\|x\|)\|x\|, \quad \forall x \in E,$$

where γ is a real function of a nonnegative argument t, $\lim_{t\to+\infty} \gamma(t) = +\infty$. Then, if the operator $\Phi(x) = F(x) + \varphi$ satisfies in E the condition of pseudomonotonicity, the equation

$$F(x) + \varphi = \theta$$

has at least one solution $x_0 \in E$ (here $\varphi \in E^*$).

It is clear that every monotone operator will also be pseudomonotone. For checking the pseudomonotonicity condition, the following lemmas, valid for real functions, may be useful.

Lemma 3. Let a continuous function $F : R^1 \to R^1$ be nondecreasing on the sets $I_1 = (-\infty, a)$, $I_3 = (b, +\infty)$, and on the set $I_2 = [a, b]$ satisfy the conditions

$$\sup_{x\in[a,b]} F(x) \leq 2 \inf_{x\in[a,b]} F(x), \qquad \inf_{x\in[a,b]} F(x) > 0.$$

Then for any $c \geq 0$ the function $\Phi(x) = F(x) + c$ is pseudomonotone in R^1.

Lemma 4. If $F : R^1 \to R^1$ is continuous and such that

$$\inf_{x\in R^1} F(x) > 0, \qquad \sup_{x\in R^1} F(x) \leq 2 \inf_{x\in R^1} F(x),$$

then F is a pseudomonotone function in R^1.

We also note that adding a negative number to a function may lead to the loss of pseudomonotonicity. For example, the functions $(\sin x + 4)$ and $(x^3 - 3x + 7)$ are pseudomonotone in R^1, but $\sin x$ and $(x^3 - 3x + 2)$ are not.

All-Union Correspondence
Electrical Engineering Institute of Communications

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1. A. N. Tikhonov, *Math. Ann.*, **111**, 767 (1935).
2. M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, 1956.
3. R. I. Kachurovskii, DAN, **173**, No. 3 (1967).
4. R. I. Kachurovskii, *Inform. Bull.* No. 7, International Congress of Mathematicians, Moscow, 1966.
5. F. E. Browder, *Proc. Nat. Acad. Sci. U. S. A.*, **53**, No. 6 (1965).
6. F. E. Browder, *Bull. Am. Math. Soc.*, **69**, No. 6, 862 (1963).
7. F. E. Browder, *Proc. Nat. Acad. Sci. U. S. A.*, **56**, No. 2, 419 (1966).

Note: Figure translations are in progress. See original paper for figures.