ON THE SOLVABILITY OF THE FIRST BOUNDARY-VALUE PROBLEM FOR A SECOND-ORDER ELLIPTIC EQUATION WITH DIVERGENT PRINCIPAL PART IN \(W_2^1\)

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Abstract

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ON THE SOLVABILITY OF THE FIRST BOUNDARY-VALUE PROBLEM FOR A SECOND-ORDER ELLIPTIC EQUATION WITH DIVERGENT PRINCIPAL PART IN W_2^1

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In the paper ([1]), for the uniformly elliptic equation

$$Lu \equiv \frac{\partial}{\partial x_i}\left(a_{ij}u_{x_j} + a_i u\right) + b_i u_{x_i} + au = f \tag{1}$$

in the case when $a_i^2,\ b_i^2,\ a \in L_p$, and $f \in L_{2n/(n+2)}$ for $p > n/2$ $(n \geq 3)$, conditions are given which ensure the existence and uniqueness of a generalized solution from W_2^1 of the first boundary-value problem for equation (1). In the present paper, conditions are given which ensure the existence and uniqueness of a generalized solution from W_2^1 for $p \leq n/2$.

First the problem is considered for the case when $a_i^2,\ b_i^2,\ a$ belong to spaces $L_{(r_1,r_2)}$ with mixed norm. We shall denote each point x of n-dimensional Euclidean space R^n in the form of the pair $(\bar{x}_s, \bar{x}_{n-s})$, where $\bar{x}_s(x_1, x_2, \dots, x_s)$, $\bar{x}_{n-s}(x_{s+1}, \dots, x_n)$. By R^s (R^{n-s}) we denote the s-dimensional ($(n-s)$-dimensional) space of points $\bar{x}_s$ $(\bar{x}_{n-s})$, $1 \leq s \leq n$. Let Ω_s be an s-dimensional bounded domain in R^s; Ω_{n-s} an $(n-s)$-dimensional bounded domain in R^{n-s}, and $\Omega = \Omega_s \times \Omega_{n-s}$. The space $L_{(r_1,r_2)}(\Omega_s, \Omega_{n-s})$ is defined as the set of functions $f(x)$, defined in Ω, for which the norm is bounded

$$\|f\|_{L_{(r_1,r_2)}(\Omega_s,\Omega_{n-s})} = \Big\| \|f(x_s, x_{n-s})\|_{L_{r_1}(\Omega_s)} \Big\|_{L_{r_2}(\Omega_{n-s})},$$

where

$$\|\cdot\|_{L_{r_i}(D)} = \begin{cases} \left(\int_D |\cdot|^{r_i}\, d\omega^i\right)^{1/r_i}, & \text{if } 1 \leq r_i < \infty, \\ \operatorname{vrai\,max}_D |\cdot|, & \text{if } r_i = \infty. \end{cases}$$

Lemma 1. *If the positive numbers r_1, r_2, q_1, q_2 satisfy at least one of the following three conditions: a) $r_1 < q_1,\ q_2 < r_2$; b) $r_2 < q_1,\ q_2 < r_1$; c) (r_1, r_2), (q_1, q_2) are distinct solutions of the equation $2x_1x_2 - (n-s)x_1 - sx_2 - \alpha = 0,\ \alpha > 0$, then*

$$(L_{(r_1,r_2)}(\Omega) \cup L_{(q_1,q_2)}(\Omega)) \setminus L_{(r_1,r_2)}(\Omega) \neq \emptyset, \quad (L_{(r_1,r_2)}(\Omega) \cup L_{(q_1,q_2)}(\Omega)) \setminus L_{(q_1,q_2)}(\Omega) \neq \emptyset,$$

$\emptyset$ is the empty set.

We denote by X^α the following class of Banach spaces (α is any positive number):

$$X^\alpha = \left\{ L_{(r_1,r_2)}(\Omega), \quad \begin{matrix} 2r_1r_2 - (n-s)r_1 - sr_2 - \alpha = 0, \quad \alpha > 0, \\ \infty > r_1 > \begin{cases} 1, & \text{if } s = 1, \\ s/2, & \text{if } s \geq 2, \end{cases} \\ \infty > r_2 > \begin{cases} 1, & \text{if } n - s = 1, \\ (n-s)/2, & \text{if } n - s \geq 2. \end{cases} \end{matrix} \right\}.$$

We note that any two spaces $L_{(r_1,r_2)}(\Omega)$, $L_{(\bar{r}_1,\bar{r}_2)}(\Omega)$, belonging to X^α, are distinct if only $r_1 \neq \bar{r}_1$ and $r_2 \neq \bar{r}_2$.

By D_α we denote the set of those points (r_1, r_2) of the plane r_1Or_2 for which $L_{(r_1,r_2)}(\Omega) \subset X^\alpha$.

Theorem 1. If $p_i = 2r_i'$ $(i = 1, 2)$, $1/r_i + 1/r_i' = 1$, $(r_1, r_2) \in D_\alpha$, then

$$\|v\|_{L_2(\Omega)} \leq c_0(\operatorname{mes}\Omega_{n-s})^{1/\rho - 1/q} \|\nabla v\|_{L_2(\Omega)}, \qquad \forall v \in \dot{W}_2^1(\Omega); \tag{2}$$

$$\|u\|_{L_{(p_1,p_2)}(\Omega)} \leq c\|u\|_{W_2^1(\Omega)}, \qquad \forall u \in W_2^1(\Omega), \tag{3}$$

$$\|v\|_{L_{(p_1,p_2)}(\Omega)} \leq c_1 \|\nabla v\|_{L_2(\Omega)}, \qquad \forall v \in \dot{W}_2^1(\Omega); \tag{4}$$

$$\|v\|^2_{L_{(p_1,p_2)}(\Omega)} \leq \tilde{c}\,[\varepsilon\beta\|\nabla v\|^2_{L_2(\Omega)} + \varepsilon^{-1/(1-\beta)}(1-\beta)\|v\|^2_{L_2(\Omega)}], \qquad \forall v \in \dot{W}_2^1(\Omega), \tag{5}$$

where ε is any positive number; the numbers β, ρ, and q are such that $0 < \beta < 1$; $1 < \rho < 2 < q$, $1/2r_2 < 1/\rho - 1/q < 1/2(n-s)$; the constants c, c_1 depend on Ω, while $c_0, \tilde{c}$ do not depend on Ω; Ω_i $(i = s, n-s)$ satisfy the cone condition.

Let (r_1, r_2) be a fixed point belonging to D_α. We shall assume that the coefficients and the free term of equation (1) satisfy the conditions

$$\nu\xi_i\xi_i \le a_{ij}\xi_i\xi_j \le \mu\xi_i\xi_i, \qquad \nu,\mu = \text{const} > 0; \tag{6}$$

$$\|a_i^2, b_i^2, a\|_{L_{(r_1,r_2)}(\Omega)} \le \mu; \tag{7}$$

$$\|f\|_{L_{(p_1',p_2')}(\Omega)} < \infty. \tag{8}$$

We denote

$$a_0 = (\operatorname{mes}\Omega)^{-1}\int_\Omega a(x)\,dx, \qquad a(x) = a^+(x) - a^-(x), \tag{9}$$

where

$$a^+(x) = \max\{a(x) - a_0; 0\}, \qquad a^-(x) = -a_0 + \max\{-a(x) + a_0; 0\},$$

$$M \equiv \max\{\|(b_i - a_i)^2\|_{L_{(r_1,r_2)}(\Omega)},\ \|a^+\|_{L_{(r_1,r_2)}(\Omega)}\},$$

$$c_2 = 2Mc^2\nu^{-2}(1+2\nu)(\beta^{-1}-1)\beta^{1/(1-\beta)}.$$

A generalized solution from $W_2^1(\Omega)$ of the first boundary-value problem for equation (1) is a function $u(x)$ belonging to $W_2^1(\Omega)$ and satisfying the identity

$$L(u,\eta) \equiv \int_\Omega \left[(a_{ij}u_{x_j} + a_i u)\eta_{x_i} - (b_i u_{x_i} + au)\eta\right]\,dx = -\int_\Omega f\eta\,dx$$

for every $\eta(x)$ from $W_2^1(\Omega)$, and the condition

$$u(x) - \varphi(x) \in \dot{W}_2^1(\Omega),$$

where $\varphi(x)$ is an extension from the boundary Γ to the whole domain Ω of the function $\tilde{\varphi}(z)$ determining the boundary values of $u(x)$, i.e.

$$u|_\Gamma = \varphi|_\Gamma. \tag{10}$$

Theorem 2. Let $(r_1, r_2) \subset D_\alpha$. If conditions (6), (7) and

$$\left(2c_2 + \frac{8}{\nu}a_0\right)c_0^2(\operatorname{mes}\Omega_{n-s})^{2/\rho-2/q} < 1, \tag{11}$$

are fulfilled, then problem (1), (10) has at most one generalized solution from $W_2^1(\Omega)$.

Corollary 1. For any differential operator L, defined in Ω and satisfying conditions (6) and (7), the uniqueness theorem for the Dirichlet problem is valid in any subdomain $\Omega' = \Omega'_s \times \Omega'_{n-s}$ of the domain Ω, provided only that $\operatorname{mes}\Omega$ is sufficiently small.

Corollary 2. In domains of arbitrary dimension the uniqueness theorem for the Dirichlet problem holds for the operators $L - \lambda E$, $\lambda \geqslant \lambda_0$, if L satisfies conditions (6) and (7), and λ_0 is sufficiently large.

Corollary 3. Theorem 2 is valid for any point $(r_1, r_2) \in D_\alpha$.

Corollary 4. Theorem 2 is valid for any $\alpha > 0$.

Theorem 3. Let $(r_1, r_2) \in D_\alpha$. If conditions (6), (7) are satisfied and, in addition, the estimate

$$c_2 + \frac{4}{\gamma} a_0 \leqslant 0, \tag{12}$$

holds, then problem (1), (10) has a generalized solution from $W_2^1(\Omega)$ for any f from $L_{(p'_2, p'_2)}(\Omega)$ and $\varphi(x)$ from $W_2^1(\Omega)$.

Define the sets $\widehat{X}^\alpha$ and $\widetilde{X}^\alpha$:

$$\widehat{X}^\alpha = \bigcup_{(r_1,r_2)\in D_\alpha} L_{r_1,r_2}(\Omega), \qquad \widetilde{X}^\alpha = \bigcup_{\substack{p_i = 2r'_i \\ (r_1,r_2)\in D_\alpha}} L_{(p'_1,p'_2)}(\Omega).$$

Theorem 4. If $a_i^2, b_i^2, a \in \widehat{X}^\alpha$ and

$$\|a_i^2\|_{L_{(r_1^i, r_2^i)}(\Omega)}, \qquad \|b_i^2\|_{L_{(\tilde{r}_1^i, \tilde{r}_2^i)}(\Omega)}, \qquad \|a\|_{L_{\simeq(r_1,r_2)}(\Omega)} \leqslant \mu,$$

then, under conditions (6) and (11), problem (1), (10) has at most one generalized solution from $W_2^1(\Omega)$.

Theorem 5. Under the conditions of the preceding theorem, problem (1), (10) has a generalized solution u from $W_2^1(\Omega)$ for any $f \in \widetilde{X}^\alpha$ and φ from $W_2^1(\Omega)$.

Let

$$\widehat{X}_\alpha^p = \bigcup_{\substack{p<r_1,r_2 \\ (r_1,r_2)\in D_\alpha}} L_{(r_1,r_2)}(\Omega), \qquad \overline{X}_p^\alpha = \left\{ L_{(r_1,r_2)}(\Omega);\ \begin{array}{l} p \leqslant r_1, r_2, \\ (r_1, r_2) \in D_\alpha \end{array} \right\}.$$

Lemma 2. $\widehat{X}_p^\alpha \subset L_p(\Omega)$.

From Theorem 4 and Lemma 2 it follows:

Theorem 4′**.** Let $a_i^2, b_i^2, a \in L_p$, $p \geqslant 1$. If $a_i^2, b_i^2, a \in \widehat{X}_p^\alpha$ and

$$\|a_i^2\|_{L_{(r_1^i, r_2^i)}(\Omega)}, \qquad \|b_i^2\|_{L_{(\tilde r_1^i, \tilde r_2^i)}(\Omega)}, \qquad \|a\|_{L_{\simeq(r_1, r_2)}(\Omega)} \leqslant \mu,$$

where

$$L_{(r_1^i, r_2^i)}(\Omega), \qquad L_{(\tilde r_1^i, \tilde r_2^i)}(\Omega), \qquad L_{\simeq(r_1, r_2)}(\Omega) \in \overline{X}_p^\alpha,$$

then, under conditions (6) and (11), problem (1), (10) cannot have more than one generalized solution.

Theorem 5′**.** Under the conditions of the preceding theorem, problem (1), (10) has a generalized solution u from $W_2^1(\Omega)$ for any f from $\widetilde{X}^\alpha$ and φ from $W_2^1(\Omega)$.

Denote

$$I_1(v, \xi) = \int_\Omega (a_i v \eta_{x_i} - b_i v_{x_i} \eta - a^+ v \eta)\, dx, \qquad [u, v] = \int_\Omega (a_{ij} u_{x_i} \overline{v}_{x_j} + \overline{a} u \overline{v})\, dx,$$

$$l_\lambda(\eta) = -L(\varphi, \eta) - (f, \eta) - \lambda(\varphi, \eta). \tag{13}$$

We define the operators A and B by the equalities

$$[Av, \eta] = I_1(v, \eta), \qquad [Bv, \eta] = (v, \eta). \tag{14}$$

Theorem 6. Suppose that conditions (6), (7), and (12) are satisfied; then the problem

$$Lu - \lambda u = f, \tag{15}$$

$$u|_\Gamma = \varphi|_\Gamma \tag{16}$$

has a unique generalized solution in $W_2^1(\Omega)$ for arbitrary φ in $W_2^1(\Omega)$ and f in L_p, for all λ in the complex plane except for a countable set $\lambda = \lambda_k,\ k = 1, 2, \ldots$.

To each $\lambda = \lambda_k$ there corresponds a finite number of linearly independent solutions in $W_2^1(\Omega)$ of the equation

$$Lv - \lambda v = 0. \tag{17}$$

For $\lambda = \lambda_k$, $k = 1, 2, ...$, problem (15), (16) has a solution if and only if the conditions

$$l_{\lambda_k}(\omega_k^i) = 0, \qquad i = 1, 2, \dots, N_k, \tag{18}$$

are satisfied, where ω_k^i are all the solutions in W_2^1 of the equation

$$(E + A^*)\omega + \overline{\lambda}_k B\omega = 0 \tag{19}$$

adjoint to the equation

$$v + \lambda(E + A)^{-1}Bv = 0. \tag{20}$$

The number of conditions (18) coincides with the number of linearly independent solutions of equation (20) for $\lambda = \lambda_k$.

Theorem 7. Suppose that for all operators L^m, $m = 1, 2, ...$, of the form (1), the conditions of Theorem 3 are satisfied with the same constants. Suppose that $a_{ij}^m(x)$, while remaining uniformly bounded, converge almost everywhere to $a_{ij}(x)$, and that the functions a_i^m, b_i^m, a^m, f^m, φ^m converge to a_i, b_i, a, f, φ in the norms $L_{(2r_1,2r_2)}$, $L_{(2r_1,2r_2)}$, $L_{(r_1,r_2)}$, $L_{(p_1',p_2')}$, W_2^1, respectively.

Then the generalized solutions u^m in $W_2^1(\Omega)$ of the problems

$$L^m u = f^m, \qquad u|_\Gamma = \varphi^m|_\Gamma \tag{21}$$

converge strongly in $W_2^1(\Omega)$ to the generalized solution in $W_2^1(\Omega)$ of the limiting problem (1), (10).

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CITED LITERATURE

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