ON THE ACCURACY OF THE APPROXIMATION OF THE DISTRIBUTION OF A SUM OF INDEPENDENT RANDOM VARIABLES TO THE NORMAL DISTRIBUTION

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Abstract

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ON THE ACCURACY OF THE APPROXIMATION OF THE DISTRIBUTION OF A SUM OF INDEPENDENT RANDOM VARIABLES TO THE NORMAL DISTRIBUTION

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1. Consider a sequence of independent identically distributed random variables $X_1, \dots, X_n, \dots$ with common distribution function $V(x)$, characteristic function $v(t)$, and positive variance $\sigma^2 = DX_n < \infty$. Without loss of generality one may assume that $\mathbf{E}X_n = 0$. Denote by $F_n(x)$ the distribution function of the normalized sum

$$\frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{n} X_j,$$

and by $\Phi(x)$ the normal distribution function with zero mean and unit variance.

It is known that $\sup_x |F_n(x) - \Phi(x)| \to 0$ as $n \to \infty$. The asymptotic behavior of $F_n(x) - \Phi(x)$ as $n \to \infty$ has been studied by many authors. Under the assumption of the existence of the third moment $a_3 = \mathbf{E}X_n^3$, it follows from the known asymptotic expansions of the function $F_n(x)$ ([1]) that

$$\sup_x |F_n(x) - \Phi(x)| = n^{-1/2}(A + o(1)) \qquad (n \to \infty), \tag{1}$$

where $A = |a_3|/6\sqrt{2\pi}\sigma^3$ if X_n has a nonlattice distribution, and $A = |a_3|/6\sqrt{2\pi}\sigma^3 + h/2\sqrt{2\pi}\sigma$ if X_n has a lattice distribution with maximal span h.

We shall consider the case where $\mathbf{E}|X_n|^3 = \infty$. If, in addition, the condition $\limsup_{|t|\to\infty} |v(t)| < 1$ (Cramér' s condition (C)) is satisfied, then we obtain the relation $\sup_x |F_n(x) - \Phi(x)| \asymp \psi_{n,2}$, where the quantity $\psi_{n,2}$ is a certain functional of the function $nV(x\sigma\sqrt{n})$ (Theorem 1)*. In the case of lattice distributions the relation

$$\sup_x |F_n(x) - \Phi(x)| \asymp \left(\psi_{n,2} + \frac{1}{\sqrt{n}}\right)$$

holds (Theorem 2). In Theorems 3 and 4 analogous results are obtained for the remainder term in the asymptotic expansion of the function $F_n(x)$.

The methods of the present paper are those of papers ([1],[2]).

2. We formulate the main results. Put

$$\psi_{n,2} = \frac{1}{\sigma^2}\int_{|x|>\sigma\sqrt{n}} x^2\,dV(x) + \frac{1}{\sigma^3\sqrt{n}}\left|\int_{|x|\le\sigma\sqrt{n}} x^3\,dV(x)\right| + \frac{1}{\sigma^4 n}\int_{|x|\le\sigma\sqrt{n}} x^4\,dV(x).$$

Theorem 1. If $\mathbf{E}|X_n|^3 = \infty$ and $\limsup_{|t|\to\infty} |v(t)| < 1$, then

$$\sup_x |F_n(x) - \Phi(x)| \asymp \psi_{n,2}. \tag{2}$$

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* The relation $a_n \asymp b_n$ means that the sequences a_n and b_n satisfy the relation $0 < \liminf_{n\to\infty} a_n/b_n \le \limsup_{n\to\infty} a_n/b_n < \infty$.

Theorem 2. If X_n has a lattice distribution, then

$$\sup_x |F_n(x) - \Phi(x)| \preccurlyeq \left(\psi_{n,2} + 1/\sqrt{n}\right). \tag{3}$$

Let us note that the assertion of Theorem 2 in the case when $\mathbf{E}|X_n|^3 < \infty$ follows from the stronger relation (1).

Suppose now that $\mathbf{E}|X_n|^3 < \infty$ for some integer $k \ge 3$. Introduce the notation:

$$\Lambda_{n,\nu} = \frac{1}{\sigma^\nu n^{(\nu-2)/2}}\int_{|x|>\sigma\sqrt{n}} x^\nu\,dV(x) \quad (\nu = 1, \dots, k),$$

$$L_{n,\nu} = \frac{1}{\sigma^\nu n^{(\nu-2)/2}}\int_{|x|<\sigma\sqrt{n}} x^\nu\,dV(x) \quad (\nu = 1, 2, \dots),$$

$$\psi_{n,k} = \begin{cases} \Lambda_{n,k} + |L_{n,k+1}| + L_{n,k+2}, & \text{if } k \text{ is even,} \\ \Lambda_{n,k-1} + |\Lambda_{n,k}| + L_{n,k+1}, & \text{if } k \text{ is odd.} \end{cases}$$

Theorem 3. Let $\limsup_{|t|\to\infty} |v(t)| < 1$, and let there exist an integer $k \ge 3$ such that $\mathbf{E}|X_n|^k < \infty$, $\mathbf{E}|X_n|^{k+1} = \infty$. Then, for odd k,

$$\sup_x \left| F_n(x) - \Phi(x) - \sum_{\nu=1}^{k-2} \frac{P_\nu(-\Phi)}{n^{\nu/2}} \right| \preccurlyeq \psi_{n,k},$$

and, for even k,

$$\sup_x \left| F_n(x) - \Phi(x) - \sum_{\nu=1}^{k-2} \frac{P_\nu(-\Phi)}{n^{\nu/2}} - \frac{P'_{k-1}(-\Phi)}{n^{(k-1)/2}} \right| \preccurlyeq \psi_{n,k}.$$

Here $P_\nu(-\Phi)$ are the functions known in the theory of probabilistic asymptotic expansions (see ([3])). Namely,

$$P_\nu(-\Phi) = \sum \prod_{m=1}^{\nu} \frac{1}{r_m!} \left(\frac{\gamma_{m+2}}{\sigma^{m+2}(m+2)!} \right)^{r_m} W_{3r+\cdots+(\nu+2)r_\nu}(x), \tag{4}$$

where the summation is over all nonnegative integer solutions of the equation $r_1 + 2r_2 + \cdots + \nu r_\nu = \nu$, γ_ν is the cumulant of order ν of the random variable X_n,

$$W_s(x) = -\frac{s!}{\sqrt{2\pi}} e^{-x^2/2} \sum_{m=0}^{[s/2]} \frac{(-1)^m x^{s-2m}}{m!(s-2m)!2^m}.$$

The cumulants of the random variable X_n are expressed in the following way through its moments $\alpha_m = \mathbf{E}X_n^m$:

$$\gamma_\nu = \nu! \sum (-1)^{r_1+\cdots+r_\nu-1}(r_1+\cdots+r_\nu-1)! \prod_{m=1}^{\nu} \frac{1}{r_m!} \left(\frac{\alpha_m}{m!}\right)^{r_m}, \tag{5}$$

where the summation is over all nonnegative integers $r_1, r_2, \dots, r_\nu$ satisfying the equation $r_1 + 2r_2 + \cdots + \nu r_\nu = \nu$. Under the conditions of Theorem 3, γ_ν are defined only for $\nu \le k$, and the functions $P_\nu(-\Phi)$ only for $\nu \le k-2$. For large values of ν we shall regard (4) and (5) as formal equalities. We set the function $P'_{k-1}(-\Phi)$ equal to the sum (4) for $\nu = k-1$, in which, from γ_{k+1}, we omit the term containing α_{k+1}.

Let us formulate an analogous result for lattice distributions. Introduce the functions $S_\nu(x)$, setting

$$S_1(x) = \sum_{m=1}^{\infty} \frac{\sin 2\pi m x}{\pi m}, \qquad S_2(x) = \sum_{m=1}^{\infty} \frac{\cos 2\pi m x}{2(\pi m)^2}, \dots$$

$$\dots, \quad S_{2l}(x) = \sum_{m=1}^{\infty} \frac{\cos 2\pi m x}{2^{2l-1}(\pi m)^{2l}}, \qquad S_{2l+1}(x) \sum_{m=1}^{\infty} \frac{\sin 2\pi m x}{2^{2l}(\pi m)^{2l+1}}, \dots$$

Next, let

$$\Pi_{n,2}(x) = \Phi(x), \qquad \Pi_{n,l}(x) = \Phi(x) + \sum_{\nu=1}^{l-2} \frac{P_\nu(-\Phi)}{n^{\nu/2}} \qquad (l = 3, \dots, k),$$

$$\delta_\nu = \begin{cases} 1, & \text{if } \nu = 4m+1,\ 4m+2, \\ -1, & \text{if } \nu = 4m+3,\ 4m. \end{cases}$$

Theorem 4. Suppose X_n assumes, with positive probabilities, only values of the form $a+sh$ ($s = 0, \pm 1, \dots$), where a is some real number and h is the maximal span of the distribution. If $\mathbf{E}|X_n|^k < \infty$ for some integer $k \geq 3$, then

$$\sup_x \Big| F_n(x) - \Pi_{n,k}(x) -$$

$$- \sum_{\nu=1}^{k-2} \delta_\nu \left(\frac{h}{\sigma\sqrt{n}}\right)^\nu S_\nu\left(\frac{x\sigma\sqrt{n}}{h} - \frac{na}{h} + \left[\frac{na}{h}\right]\right) \frac{d^\nu}{dx^\nu} \Pi_{n,k-\nu}(x) \Big| \succ \left(\psi_{n,k} + n^{-(k-1)/2}\right).$$

In the case where $\mathbf{E}|X_n|^{k+1} < \infty$, Theorem 4 follows from the known estimate of the remainder term in the asymptotic expansion of $F_n(x)$ ([1]).

3. We outline the proof of Theorems 1 and 2. By $C_1, C_2, \dots$ we shall denote certain positive constants not depending on n. Put

$$\Delta_n = \sup_x |F_n(x) - \Phi(x)|.$$

The proof of the upper estimate for Δ_n in Theorem 1 is based on applying the theorem of C. G. Esseen ([1], p. 32) with $F(x) = F_n(x)$, $G(x) = \Phi(x)$, $T = n$, and on the following expansion of the logarithm of the function $f_n(t) = v^n(t/\sigma\sqrt{n})$:

$$\ln f_n(t) = -t^2/2 + (t^2 + t^4) O(\psi_{n,2} + 1/n) \qquad (n \to \infty)$$

uniformly in t in the interval $|t| < \sqrt{n}$. The proof of the upper estimate for Δ_n in Theorem 2 is carried out analogously; in this case we put $T = C_1\sqrt{n}$.

In proving the lower estimates for Δ_n, the methods of the work of I. A. Ibragimov ([2]) are used. Consider bounded functions $A(x)$ and $B(t)$ such that

$$\int |A(x)|\, dx < \infty, \qquad B(t) = \int e^{itx} A(x)\, dx.$$

It is not difficult to see that the functions $F_n(x)-\Phi(x)$ and $(f_n(t)-e^{-t^2/2})/-it$ belong to $L_2(-\infty,\infty)$ and constitute a Fourier-transform pair. According to Parseval' s identity, we have

$$-2\pi\int(F_n(x)-\Phi(x))\overline{A(x)}\,dx=\int(f_n(t)-e^{-t^2/2})\frac{\overline{B(t)}}{it}\,dt.$$

Consequently,

$$\Delta_n\succ C_2\left|\int(f_n(t)-e^{-t^2/2})\frac{\overline{B(t)}}{it}\,dt\right|. \tag{6}$$

Further, uniformly with respect to t in the interval $|t|<T_n=\min(\psi_{n,2}^{-1/4};n^{1/4})$,

$$f_n(t)-e^{-t^2/2}=e^{-t^2/2}\left[n\left(v\left(\frac{t}{\sigma\sqrt{n}}\right)-1+\frac{t^2}{2n}\right)+(t^4+t^8)O\left(\psi_{n,2}^2+\frac{1}{n}\right)\right].$$

Let $|B(t)|<C_3e^{-t^2/4}$. Then the integral on the right-hand side of (6) is equal to

$$n\int\frac{\overline{B(t)}}{it}e^{-t^2/2}\left(v\left(\frac{t}{\sigma\sqrt{n}}\right)-1+\frac{t^2}{2n}\right)\,dt+O\left(\psi_{n,2}^2+\frac{1}{n}\right)=$$

$$=n\iint\frac{\overline{B(t)}}{it}e^{-t^2/2}\left(e^{itx/\sigma\sqrt{n}}-1+\frac{t^2x^2}{2\sigma^2n}\right)\,dt\,dV(x)+O\left(\psi_{n,2}^2+\frac{1}{n}\right)=$$

$$=I+O\left(\psi_{n,2}^2+\frac{1}{n}\right).$$

Putting here $B(t)=-ite^{-t^2/2}$, we find

$$I=n\sqrt{\pi}\int\left(e^{-x^2/4\sigma^2n}-1+\frac{x^2}{4\sigma^2n}\right)\,dV(x)>C_4(\Lambda_{n,2}+L_{n,4});$$

for $B(t)=t^2e^{-t^2/2}$ we obtain

$$|I|=n\sqrt{\pi}\left|\int\frac{x}{2\sigma\sqrt{n}}e^{-x^2/4\sigma^2n}\,dV(x)\right|>\sqrt{\pi}\left(\frac{1}{8}|L_{n,3}|-\Lambda_{n,2}-L_{n,4}\right).$$

Hence it follows that

$$\Delta_n > C_5\psi_{n,2} + O(1/n). \tag{7}$$

Under the conditions of Theorem 1, $n\psi_{n,2} \to \infty$ as $n \to \infty$, and, consequently, we have $\Delta_n > C_6\psi_{n,2}$. From this (2) follows.

Suppose now that X_n has a lattice distribution with maximal span h. Putting in (6) $B(t) = \exp[-\frac{1}{2}(t + \frac{2\pi}{h}\sigma\sqrt{n})^2]$, we easily find that

$$\Delta_n \geq C_2 \left| \int_{|t|<T_n} \frac{f_n(t)e^{-t^2/2}}{t - \frac{2\pi}{h}\sigma\sqrt{n}}\, dt \right| + O\left(\psi_{n,2} + \frac{1}{\sqrt{n}}\right) > \frac{C_7}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right).$$

The last inequality and inequality (7) complete the proof of relation (3).

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Note: Figure translations are in progress. See original paper for figures.