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Abstract

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MATHEMATICS

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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF SYSTEMS OF INTEGRAL EQUATIONS ON THE HALF-AXIS

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Introduction. Consider the system of integral equations

$$y(x) = f(x) + \int_0^\infty k(x,s)y(s)\,ds. \tag{1}$$

Here $y(x) = \{y_1(x),\, y_2(x), ...\}$ and $f(x) = \{f_1(x),\, f_2(x), ...\}$ are vectors; $k(x,s)$ is the matrix $k_{ij}(x,s)$, with $f(x) \equiv 0$ for $x \geq x_0$, $k(x,s) = 0$ for $x - s > a$.

In the paper it is shown that the solutions of system (1), for a rather general class of kernels $k(x,s)$, have a common asymptotic behavior as $x \to \infty$ (up to a constant factor) for various functions $f(x)$. This fact is a generalization of the well-known property of solutions of integral equations with kernels depending on the difference of the arguments, when an explicit form of the solution $y(x)$ can be found.

Many equations of mathematical physics, in particular problems connected with the transport equation (Milne's problem, etc.), lead to equations of the form (1), but with $f(x) \to 0$ as $x \to \infty$ and with kernels that do not vanish for $x - s > a$, but decrease rapidly as $x - s \to +\infty$.

§ 1. **Auxiliary formulas.** Everywhere in what follows we shall assume that the solution of equation (1) can be obtained by the method of successive approximations. In addition, we shall assume that $k(x,s) \geq 0$ (we shall write $a \geq b$ when $a_{ij} \geq b_{ij}$) and $y(x) > 0$ for $x \geq x_0$, if $f(x) > 0$. Introduce the notation: $x_m = x_0 + ma$, $u_m = [x_m, x_{m+1}]$, $G_m(x,s)$ is the resolvent of the equation

$$y(x) = f(x) + \int_{x_m}^\infty k(x,s)y(s)\,ds,$$

considered for $x \geq x_m$. Writing equation (1) for $x \geq x_m$ in the form

$$y(x) = f_m(x) + \int_{x_m}^{\infty} k(x,s)y(s)\,ds,$$

$$f_m(x) = \begin{cases} \displaystyle\int_{x-a}^{x_m} k(x,s)y(s)\,ds, & \text{for } x \in u_m, \\ 0, & \text{for } x > x_{m+1}, \end{cases}$$

we obtain, for $x \in u_m$,

$$y(x) = \int_{x_{m-1}}^{x_m} \psi^{(m)}(x,s)y(s)\,ds, \tag{2}$$

where

$$\psi^{(m)}(x,s) = k(x,s) + \int_{x_m}^{x_{m+1}} G_m(x,t)k(t,s)\,dt. \tag{3}$$

Let $\bar{y}(x)$ be a solution of equation (1) with $f(x)$ replaced by $\underline{f}(x)$, and let $\bar{f}(x) > 0$ for $x \leq x_0$. The function $z(x)$ with components $z_i(x) = y_i(x)/\bar{y}_i(x)$ is a solution of the equation

$$z(x) = \int_{x_{m-1}}^{x_m} \varphi^{(m)}(x,s)z(s)\,ds. \tag{4}$$

Here $\varphi^{(m)}(x,s)$ is the matrix with components

$$\varphi_{ij}^{(m)}(x,s) = \psi_{ij}^{(m)}(x,s)\bar{y}_j(s)/\bar{y}_i(x). \tag{5}$$

In what follows we shall need the obvious equality

$$\sum_{j=1}^{n} \int_{x_{m-1}}^{x_m} \varphi_{ij}^{(m)}(x,s)\,ds = 1. \tag{6}$$

§ 2. Asymptotic properties of solutions of integral equations

1. We shall show that, under fairly general conditions, $z_i(x) \to c$ as $x \to \infty$, where c is some constant independent of i.

Theorem 1. Let

$$P_m = \max_i \sup_{x \in u_m} z_i(x) \quad \text{and} \quad p_m = \min_i \inf_{x \in u_m} z_i(x).$$

Suppose there exist finite limits

$$P = \lim_{m \to \infty} P_m \quad \text{and} \quad p = \lim_{m \to \infty} p_m.$$

If, moreover,

$$\prod_{k=1}^{\infty} (1 - \delta_k) = 0,$$

where

$$\delta_m = \min_{i_1, i_2,\, x', x'',\, E_{mj}} \inf \sum_{j=1}^{n} \left[\int_{E_{mj}} \varphi_{i_1 j}^{(m)}(x', s)\, ds + \int_{\bar{E}_{mj}} \varphi_{i_2 j}^{(m)}(x'', s)\, ds \right]$$

(E_{mj} are arbitrary measurable sets, $\bar{E}_{mj}$ is the complement of E_{mj} to u_{m-1}), then $P = p$, i.e., there exists a common finite limit

$$z(\infty) = \lim_{x \to \infty} z_i(x),$$

and moreover

$$|z_i(x) - z(\infty)| \leq (P_0 - p_0) \prod_{k=1}^{m} (1 - \delta_k), \quad \text{for } x \in u_m.$$

Proof. We have

$$P_m = \max_i \sup_{x \in u_m} \left[\sum_{j=1}^{n} \int_{x_{m-1}}^{x_m} \varphi_{ij}^{(m)}(x, s) z_j(s)\, ds \right] \leq$$

$$\leq P_{m-1} \max_i \sup_{x \in u_m} \left[\sum_{j=1}^{n} \int_{x_{m-1}}^{x_m} \varphi_{ij}^{(m)}(x, s)\, ds \right] = P_{m-1}$$

by virtue of (6). Similarly, we obtain $p_m \geq p_{m-1}$. Thus the sequences $\{P_m\}$ and $\{p_m\}$ are monotone and bounded, whence the first part of the theorem follows.

Next,

$$P_m - p_m = \max_{i_1, i_2} \sup_{x', x'' \in u_m} \left[z_{i_1}(x') - z_{i_2}(x'') \right].$$

But

$$z_{i_1}(x') - z_{i_2}(x'') = \sum_{j=1}^{n} \int_{x_{m-1}}^{x_m} \left[\varphi_{i_1 j}^{(m)}(x', s) - \varphi_{i_2 j}^{(m)}(x'', s) \right] ds.$$

Let E_{mj} be the set on which $\varphi_{i_1 j}^{(m)}(x', s) \geq \varphi_{i_2 j}^{(m)}(x'', s)$. Splitting the integral over u_{m-1} into integrals over E_{mj} and $\bar{E}_{mj}$ $(E_{mj} + \bar{E}_{mj} = u_{m-1})$, we note them in the first integral $z_j(s)$ by P_{m-1}, and in the second integral by p_{m-1}. Then, by virtue of the obvious relation:

$$\sum_{j=1}^{n} \int_{E_{mj}} \left[\varphi_{i_1 j}^{(m)}(x', s) - \varphi_{i_2 j}^{(m)}(x'', s) \right] ds =$$

$$= \sum_{j=1}^{n} \int_{E_{mj}} \left[\varphi_{i_1 j}^{(m)}(x'', s) - \varphi_{i_2 j}^{(m)}(x', s) \right] ds \leqslant 1 - \delta_m$$

we obtain the inequality $P_m - p_m \leqslant (1 - \delta_m)(P_{m-1} - p_{m-1})$. From this inequality the second part of the theorem follows.

2. The expression for δ_m in Theorem 1 is rather complicated. Therefore we shall give cruder, but simpler and obvious estimates for δ_m:

$$\delta_m \geqslant \min_{i_1, i_2} \left\{ \sum_{j=1}^{n} \int_{x_{m-1}}^{x_m} \min \left[\alpha_{i_1 j}^{(m)}(s), \alpha_{i_2 j}^{(m)}(s) \right] ds \right\}, \tag{7}$$

where

$$\alpha_{ij}^{(m)}(s) = \inf_{x \in u_m} \varphi_{ij}^{(m)}(x, s).$$

If $\varphi^{(m)}(x, s) > 0$, then the estimates can be simplified still further:

$$\delta_m \geqslant \sum_{j=1}^{n} \int_{x_{m-1}}^{x_m} \left[\min_i \alpha_{ij}^{(m)}(s) \right] ds. \tag{8}$$

If the solution $\overline{y}(x)$ is not known explicitly, then it is necessary to obtain estimates that do not contain $\overline{y}(x)$. For simplicity we shall obtain such estimates under $\psi^{(m)}(x, s) > 0$ for $s > x_{m-1}$. We have

$$\alpha_{ij}(s) \geqslant \frac{\inf\limits_{x} \psi_{ij}^{(m)}(x,s) \cdot \sum\limits_{l=1}^{n} \int_{x_{m-2}}^{x_{m-1}} \overline{y}_l(t) \psi_{jl}^{(m-1)}(s,t)\, dt}{\sup\limits_{x} \left\{ \sum\limits_{l=1}^{n} \int_{x_{m-2}}^{x_{m-1}} \overline{y}_l(t)\, dt \cdot \sum\limits_{p=1}^{n} \int_{x_{m-1}}^{x_m} \psi_{ip}^{(m)}(x,z) \psi_{pl}^{(m-1)}(z,t)\, dz \right\}} \geqslant$$

$$\geqslant \frac{\left[\min\limits_{i} \inf\limits_{x} \psi_{ij}^{(m)}(x,s)\right] \left[\min\limits_{l} \inf\limits_{t} p(t) \psi_{jl}^{(m-1)}(s,t)\right]}{\left[\max\limits_{ij} \sup\limits_{xs} \psi_{ij}^{(m)}(x,s)\right] \left[\max\limits_{ij} \sup\limits_{t} p(t) \cdot \int_{x_{m-1}}^{x_m} \psi_{ij}^{(m-1)}(z,t)\, dz\right]}. \tag{9}$$

Here $p(t)$ is an arbitrary nonnegative function. We cannot put $p(t) = 1$, since

$$\inf_{t} \psi_{jl}^{(m-1)}(s,t) = \psi_{jl}^{(m-1)}(s, x_{m-2}) = 0.$$

Taking this consideration into account, one may put $p(t) = 1/(t - x_{m-2})$. Substituting (9) into (8), we obtain an estimate for δ_m that does not contain $\overline{y}(x)$.

3. Thus, we obtain that the estimate for the quantity δ_m reduces to estimating the functions $\psi^{(m)}(x,s)$ from above and below. If the functions $\psi^{(m)}(x,s)$ are known explicitly, then such estimates present no difficulty. If, however, the functions $\psi^{(m)}(x,s)$ are not known explicitly, then one may use the obvious consideration: if $\underline{k}(x,s) \leqslant k(x,s) \leqslant \tilde{k}(x,s)$, then $\underline{\psi}^{(m)}(x,s) \leqslant \psi^{(m)}(x,s) \leqslant \widetilde{\psi}^{(m)}(x,s)$ (the functions $\underline{\psi}^{(m)}(x,s)$ and $\widetilde{\psi}^{(m)}(x,s)$ are obtained with the help of the kernels $\underline{k}(x,s)$ and $\tilde{k}(x,s)$ in the same way as the function $\psi^{(m)}(x,s)$ is obtained with the help of $k(x,s)$). In this case, as the kernels $\underline{k}(x,s)$ and $\tilde{k}(x,s)$ it is convenient to take such kernels for which the functions $\underline{\psi}^{(m)}(x,s)$ and $\widetilde{\psi}^{(m)}(x,s)$ can be computed explicitly. If, in particular, $k(x,s) \geqslant \bar{c}_m$

for $x - a \leq s \leq x,\ x \in u_m$ ($\bar{c}_m$ is a constant matrix), then $\psi^{(m)}(x,s) \geq \bar{\psi}^{(m)}(x,s)$, where

$$\bar{\psi}^{(m)}(x,s) = \begin{cases} \bar{c}_m e^{\bar{c}_m (x - x_m)}, & \text{for } s > x - a, \\ \bar{c}_m \left[e^{\bar{c}_m (x - x_m)} - e^{\bar{c}_m (x - a - s)} \right], & \text{for } s \leq x - a. \end{cases}$$

If we consider Volterra equations, then to estimate the function $\psi^{(m)}(x,s)$ from above one may use a similar estimate: if $k(x,s) \leq \tilde{c}_m$ for $x - a \leq s \leq x$ ($x \in u_m$), then $\psi^{(m)}(x,s) \leq \tilde{\psi}^{(m)}(x,s)$ (the expression for $\tilde{\psi}^{(m)}(x,s)$ is obtained from the expression for $\psi^{(m)}(x,s)$ by replacing $\bar{c}_m$ by $\tilde{c}_m$).

It is easy to obtain for the function $\psi^{(m)}(x, s)$ an upper estimate also in the case of Fredholm equations with kernels satisfying the condition

$$\sum_{j=1}^{n} \int_{0}^{\infty} k_{ij}(x, s)\, ds \leq \alpha < 1.$$

In this case, to estimate the resolvent $G_m(x, s)$, it is convenient to use the obvious estimates:

$$G_m(x, s) \leq G(x, s), \qquad \sum_{j=1}^{n} \int_{0}^{\infty} G_{ij}(x, s)\, ds \leq \frac{\alpha}{1-\alpha}.$$

From the functional relation for the resolvent $G_m(x, s)$

$$G_m(x, s) = k(x, s) + \int_{x_m}^{s+a} G_m(x, t) k(t, s)\, dt$$

for $x \in u_m,\ s \in u_m$, we have

$$\max_{ij} [G_m(x, s)]_{ij} \leq \frac{1}{1-\alpha} \left[\max_{i,j} \sup_{s \in u_m,\ x_m \leq x \leq s+a} k_{ij}(x, s) \right] = \frac{q_m}{1-\alpha}.$$

Hence

$$\psi_{ij}^{(m)}(x, s) \leq q_{m-1} + \frac{q_{m-1} q_m (s - x_{m-1})}{1-\alpha} \quad \text{for } s > x - a,$$

$$\psi_{ij}^{(m)} \leq \frac{q_{m-1} q_m (s - x_{m-1})}{1-\alpha} \quad \text{for } s \leq x - a.$$

Substituting these estimates into the expression for δ_m, one can obtain conditions for the existence of a single limit

$$z(\infty) = \lim_{x \to \infty} z_i(x).$$

In view of the cumbersomeness of the estimates δ_m for the cases considered above, we shall not present them. We note only that, when the inequalities

$$0 < \bar{c} \leq k_{ij}(x, s) \leq \tilde{c}$$

($\bar{c}, \tilde{c}$ are certain constants) hold in the domain $2a \geq x - s \geq a$, for the cases considered above $\delta_m \geq \delta > 0$, and, consequently, there exists a single limit

$$z(\infty) = \lim_{x \to \infty} z_i(x).$$

§ 3. All the results obtained are easily generalized to the continuum case, when instead of equation (1) one considers equations of the form

$$y(x,\lambda) = f(x,\lambda) + \int_\Omega d\mu \int_0^\infty k(x,\lambda;s,\mu)y(s,\mu)\,ds,$$

where $f(x,\lambda) \equiv 0$ for $x \geq x_0$, $k(x,\lambda;s,\mu) = 0$ for $x - s > a$, and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ belongs to some closed domain Ω. In view of the obviousness of such a generalization, we shall not consider it in detail.

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Note: Figure translations are in progress. See original paper for figures.

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