NONUNITARY OPERATORS WITH ABSOLUTELY CONTINUOUS SPECTRUM ON THE UNIT CIRCLE

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Abstract

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MATHEMATICS

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NONUNITARY OPERATORS WITH ABSOLUTELY CONTINUOUS SPECTRUM ON THE UNIT CIRCLE

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1°. We shall say that an operator T, acting in a separable Hilbert space H, belongs to the class $\mathcal{E}$ if it is linearly similar to some unitary operator with absolutely continuous spectrum. We shall say that $A \in \mathcal{N}$ if $(A - iE)(A + iE)^{-1} \in \mathcal{E}$. Let T be a bounded operator together with its inverse. We define the characteristic matrix-function $w(\mu)$ of the operator T by the equality ([1])

$$w(0)w(\mu) = E - J\|((E - \mu T)^{-1} g_\alpha, g_\beta)\|,$$

where g_α is a channel system of vectors (see ([2–4])).

Recall that the simple part of T is the operator induced by T on the subspace

$$H_1 = \overline{\sum_{k=-\infty}^{\infty} T^k D_T}, \quad \text{where} \quad D_T = \overline{(E - T^*T)H}.$$

Theorem 1. *If the characteristic matrix-function of the operator T satisfies, for some c, the condition*

$$\|w(\mu)\| \le c, \qquad |\mu| \ne 1, \tag{1}$$

then the simple part of T is linearly similar to a unitary operator with absolutely continuous spectrum.

In the case $J = E$ and for a special choice of channel vectors, it was shown in the work of B. Nad' and Ch. Foias ([5]) that inequality (1) is a necessary and sufficient condition for the similarity of the operator T to some unitary operator (see also ([6–7])).

It is sometimes more convenient to use Theorem 1 by writing $w(\mu)$ in operator form (see ([3])).

There exists a bounded operator R, mapping some Hilbert space G into H, and satisfying the conditions:

1. $E - T^*T = RJR^*$, where J acts in G and $J = J^*$, $J^2 = E$.
2. The spectrum of $E - JR^*R$ is strictly positive.

By condition 2 there exists a J-Hermitian operator $w(0)$ with strictly positive spectrum such that

$$w^2(0) = E - JR^*R.$$

We define the characteristic operator-function $w(\mu)$ of the operator T in the following way:

$$w(0)w(\mu) = E - JR^*(E - \mu T)^{-1}R.$$

The values of the operator-function $w(\mu)$ are operators acting in G.

2°. Let the operator A have the form

$$A = A_R + iA_I, \tag{2}$$

where A_R and A_I are self-adjoint operators acting in H, with A_I bounded. There exists a bounded operator K, mapping some Hilbert space G into H, such that

$$A_I = KJK^*,$$

where J acts in G and $J^* = J,\ J^2 = E$.

The operator function

$$W(\lambda) = E - 2iK^*(A - \lambda E)^{-1}KJ$$

is called the characteristic function of the operator A.

For bounded dissipative operators A, the definition introduced does not coincide with that given in the paper ([3]).

We note that the values $W(\lambda)$ are operators in G. An operator A with real spectrum will be called simple if $T = (A - iE) \times (A - iE)^{-1}$ is simple.

Theorem 2. *If the characteristic function of the operator A for some c satisfies the condition*

$$\|W(\lambda)\| \leq c, \qquad \operatorname{Im}\lambda \neq 0,$$

then the simple part of A is linearly similar to a self-adjoint operator with absolutely continuous spectrum.

For $J = E$, a close fact is contained in the paper ([7]).

Let us now consider the operator

$$A = A_0 + B, \tag{3}$$

where A_0 is a self-adjoint operator, B is a bounded operator. Represent B in the form

$$B = B_1^* B_2, \tag{4}$$

where B_1 and B_2 are bounded operators.

An operator of the form (3) admits representations (2).

Put $R_0(\lambda) = (A_0 - \lambda E)^{-1}$, $Q(\lambda) = B_2 R_0(\lambda) B_1^*$.

Theorem 3. *Let the operators $Q(\lambda)$, $(E + Q(\lambda))^{-1}$, $K^* R_0(\lambda) K$, $K^* R_0(\lambda) B_1^*$, $B_2 R_0(\lambda) K$ be uniformly bounded off the real axis. Then the simple part of A is linearly similar to a self-adjoint operator with absolutely continuous spectrum.*

It is useful to compare this result with a theorem of T. Kato ([8]).

3°. Consider in the space $L^2(E_m)$ the Schrödinger operator

$$A = A_0 + q,$$

where

$$A_0 u = -\sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} u, \qquad qu = q(x)u, \qquad x \in E_m, \quad m \geq 3.$$

Introduce the notation

$$B_1 f = |q(x)|^{1/2} f(x), \qquad B_2 f = \frac{\overline{q(x)}}{|q(x)|^{1/2}} f(x), \qquad Q(\lambda) = B_2 R_0(\lambda) B_1^*.$$

Theorem 4. *Let the function $q(x)$ be bounded and $q(x) \in L^p(E_m)$ $(1 \leq p < m/2)$. If the operator function $[E + Q(\lambda)]^{-1}$ is uniformly bounded for* $\operatorname{Im} \lambda \neq 0$, *then the operator A is linearly similar to a self-adjoint operator with absolutely continuous spectrum, i.e. $A \in \mathcal{N}$.*

Corollary. *If in Theorem 4 $p = 1$, then the operators A and A_0 are linearly similar and comparable.*

Recall that the operators A and A_0 are called comparable ([1]) if there exist wave operators, bounded together with their inverses,

$$W_\pm(A, A_0) = s\text{-}\lim_{t\to\pm\infty} e^{iAt}e^{-iA_0t},$$

and, moreover,

$$AW_\pm(A, A_0) = W_\pm(A, A_0)A_0.$$

Under somewhat weaker requirements on $q(x)$, in the work of T. Kato ([8]) the linear similarity and comparability of the operators A and A_0 were proved. In that work, however, it was additionally assumed that

$$\|Q(\lambda)\| \le \rho < 1.$$

4°. Consider a differential expression of the form

$$ly = -y'' + yp(r),$$

where $y(r) = [y_1(r), y_2(r), \dots, y_m(r)]$ $(m < \infty)$ is a vector function, $p(r)$ is a matrix of order m, and

$$\sigma = \int_0^\infty \|p(r)\|\,dr < \infty.$$

The expression $l(y)$ and the condition

$$y'(0) - y(0)\theta = 0 \qquad (\theta \text{ is a matrix})$$

generate the differential operator L_θ. The non-self-adjoint operator L_θ was studied in the work of M. A. Naimark ([9]) for $m = 1$ and

$$\int_0^\infty (1 + r^2)\|\nu(r)\|\,dr < \infty.$$

Introduce the matrix $y(r, s)$, which satisfies the equation

$$l(y) - s^2y = 0, \qquad \operatorname{Im} s \ge 0,$$

and the condition

$$\lim_{r\to\infty} y(r,s)e^{-isr} = E.$$

Denote by $D(s) = y'(0,s) - y(0,s)\theta$. We shall compare the operator L_θ with the simplest operator $\mathcal{L}_0$:

$$\mathcal{L}_0 f = -d^2 f/dr^2, \qquad f(0) = 0, \qquad f \in L^2_m(0,\infty).$$

Theorem 5. *If there exists a c such that*

$$\|D^{-1}(s)\| \le c, \qquad \operatorname{Im} s \ge 0, \qquad \text{and} \quad \int_0^\infty (1 + r^{3/2})\|p(r)\|\, dr < \infty,$$

then the operators L_θ and $\mathcal{L}_0$ are linearly similar and comparable.

The conditions of Theorem 5 can be weakened if dissipative operators are considered.

Theorem 6. *Let the operator L_θ satisfy the following conditions:*

1. $$\frac{\theta - \theta^*}{i} \ge 0, \qquad \frac{p(r) - p^*(r)}{i} \ge 0.$$
2. $$\int_0^\infty \|p(r)\|\, dr < \infty.$$
3. *There exists a c such that $\|D(s)\| \le c$, $\|D^{-1}(s)\| \le c$.*

Then the operators L_θ and $\mathcal{L}_0$ are linearly similar and comparable.

5°. Let the operator A act in $L^2_m(a,b)$ $(-\infty \le a < b \le \infty)$ and be defined by the formula

$$A = A_0 + iB,$$

where $A_0 f = xf$, and B is a self-adjoint nuclear operator.

The operator B can be written in the form

$$Bf = \frac{1}{2}\sum_{\alpha,\beta}^{n} (f, g_\alpha)\, j_{\alpha\beta} g_\beta, \qquad \sum_{\alpha=1}^{n} \|g_\alpha\|^2 < \infty,$$

where $n \le \infty$, $J = \|j_{\alpha\beta}\|$ is a certain Hermitian matrix such that $J^2 = E$.

Introduce the matrices

$$G(x) = \|g_\alpha(x) g_\beta^*(x)\|_{\alpha,\beta=1}^n, \qquad V(\lambda) = \frac{1}{2}\int_a^b \frac{G(x)}{x-\lambda}\,dx.$$

Theorem 7. Let the matrix $G(x)$ satisfy the conditions:

1. The function $G(x)$ is bounded on the segment $[a, b]$, and

$$\int_a^b \|G(x)\|\,dx < \infty.$$

2. There exists an M such that, for all t and for some $c > 0$,

$$\int_{t-c}^{t+c} \left\| \frac{G(x) - G(t)}{x - t} \right\| dx \le M.$$

3. $\overline{\lim}\, \|G(x)\| \ln(x - a) < \infty, \quad x \to a.$
4. $\overline{\lim}\, \|G(x)\| \ln(b - x) < \infty, \quad x \to b.$

(If $a = -\infty$ or $b = \infty$, then respectively condition 3 or 4 is omitted.)

If the matrix-function $[E + iV(\lambda)J]^{-1}$ is uniformly bounded for $\operatorname{Im}\lambda \neq 0$, then the operators A and A_0 are linearly similar and comparable.

6°. Consider the operator

$$Af = xf + i\int_a^x f(t)\beta(t)J\,dt\,\beta(x) \qquad (-\infty < a < b < \infty),$$

where $\beta(t)$ is a nonnegative matrix of order m ($m \le \infty$), and $J = J^*$ and $J^2 = E$. A broad class of operators (2) is reduced to such triangular form.

Theorem 8. Let the following conditions be fulfilled:

1. There exists a matrix $U(t)$ such that

$$\beta^2(t)J = U^*(t)H(t)U^{-1}(t),$$

where $H(t)$ is a self-adjoint matrix.

2. The matrices $U(t)$, $U^{-1}(t)$, and $H(t)$ are uniformly bounded on the segment $[a, b]$.
3. The inequality holds

$$\|U(t_2) - U(t_1)\| \leq K|t_2 - t_1|, \qquad t_1, t_2 \in [a, b],$$

where K is some constant.

Then the simple part of A is linearly similar to some self-adjoint operator with absolutely continuous spectrum.

We note that for $J = E$ one may put $U(t) = E$ (see (6)).

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Note: Figure translations are in progress. See original paper for figures.