PROJECTIVE SETS OF TOPOLOGICAL SPACES OF WEIGHT τ

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PROJECTIVE SETS OF TOPOLOGICAL SPACES OF WEIGHT τ

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Let I be a space of indices of cardinality $\tau = \aleph_\nu$, which we shall regard as a strongly inaccessible cardinal number ([1]). We study the projective hierarchies of classes of sets of topological spaces D^τ, J^τ and topological τ-spaces $D^{\omega_\nu}, J^{\omega_\nu}$ ([3]). We shall show that the classes of these hierarchies are determined by substantially different operations, depending on the topology of the space.

The operation

$$\Psi^x_{N(x)}\{E_i\} = \bigcup_x (\{x\} \cap \Psi^\circ_{N(x)}\{E_i\})$$

is called a set-theoretic operation with variable base. A class $P \subset \mathfrak{P}\mathfrak{R}_{xy} = \mathfrak{P}(\mathfrak{R}_x \times \mathfrak{R}_y)$, where $\mathfrak{R}$ is the basic space, is called projective with respect to the class $\mathfrak{K} \subset \mathfrak{P}\mathfrak{R}_x$ if there exists such a variable base $L(y)$ that

$$P = \Psi^{(x,y)}_{L(y)}(\mathfrak{K}^*),$$

where $\mathfrak{K}^*$ is the class of all sets of the form $K \times \mathfrak{R}_y$ $(K \in \mathfrak{K})$,

$$\Psi^{(x,y)}_{L(y)}\{E_i\} = \Psi^{(x,y)}_{L(x,y)}\{E_i\}, \quad \text{when } L(x,y) = L(x) \quad ({}^2).$$

L. V. Kantorovich and E. M. Livenson ([2]) showed that if a class P is projective with respect to a class $\mathfrak{K}$, then for the class P' of projections of sets belonging to P the equality

$$P' = \Psi_{L_0}(\mathfrak{K})$$

holds, where

$$L_0 = \bigcup_{y \in \mathfrak{R}y} L(y).$$

Theorem 1. The classes F and G of the space $J^{\omega_\nu}_{xy}$ $(D^{\omega_\nu}_{xy}, D^\tau_{xy}, J^\tau_{xy})$ are projective, respectively, with respect to the classes F and G of the space $J^{\omega_\nu}_x$ $(D^{\omega_\nu}_x, D^\tau_x, J^\tau_x)$. The base $L(y)$ for the class $G \subset \mathfrak{P}J^{\omega_\nu}_{xy}$, where

$y = \{i_0, \dots, i_\alpha, \dots\} \in J_{xy}^{\omega_\nu}$, consists of all chains of the family $(\{\lambda(i_0, \dots, i_\alpha)\})_\alpha$; the base $L^c(y)$ consists of one chain

$$\{\lambda(i_0), \lambda(i_0, i_1), \dots, \lambda(i_0, i_1, \dots, i_\alpha), \dots\},$$

where $\lambda(i_0, \dots, i_\alpha) = \lambda$ is a one-to-one mapping of the space of all tuples W of the space of indices I onto I. The base $L_1(y)$ for the class $G \subset \mathfrak{P} J_{xy}^\tau$ consists of all chains of the family

$$(\{\lambda(i_{\alpha_1}, \dots, i_{\alpha_k})\})_{\{\alpha_1, \dots, \alpha_k\} \in I''_\omega},$$

where I''_ω is the totality of all finite tuples $\{\alpha_1, \dots, \alpha_k\}$ of the space of indices I, $\alpha_1 < \alpha_2 < \dots < \alpha_k$, and $\lambda(i_{\alpha_1}, \dots, i_{\alpha_k}) = \lambda$ is a one-to-one mapping of the space X^* of all tuples $\{i_{\alpha_1}, \dots, i_{\alpha_k}\}$, for $\{\alpha_1, \dots, \alpha_k\} \in I''_\omega$, onto I. The base $L_1^c(y)$ consists of one chain

$$(\lambda(i_{\alpha_1}, \dots, i_{\alpha_k}))_{\{\alpha_1, \dots, \alpha_k\} \in I''_\omega}.$$

The base $L'(y)$ for the class $G \subset \mathfrak{P} D_{xy}^\tau$, where $y = \{\nu_0, \dots, \nu_\alpha, \dots\} \in D_y^\tau$, consists of chains of the family

$$(\{\lambda(\nu_{\alpha_1}, \dots, \nu_{\alpha_k})\})_{\{\alpha_1, \dots, \alpha_k\} \in I''_\omega};$$

the base $L_1^c(y)$ for the class $F \subset D_{xy}^\tau$ consists of one chain

$$(\lambda(\nu_{\alpha_1}, \dots, \nu_{\alpha_k}))_{\{\alpha_1, \dots, \alpha_k\} \in I''_\omega}.$$

The base $L''(y)$ for the class $G \subset \mathfrak{P} D_{xy}^{\omega_\nu}$, where $y = \{\nu_0, \dots, \nu_\alpha, \dots\} \in D_y^{\omega_\nu}$, consists of chains of the family

$$(\{\lambda(\nu_0, \dots, \nu_\alpha)\})_\alpha.$$

The base $L''^c(y)$ for the class $F \subset \mathfrak{P} D_{xy}^{\omega_\nu}$ consists of one chain

$$(\lambda(\nu_0, \dots, \nu_\alpha))_\alpha.$$

Corollary 1. The classes of projections of sets belonging to the classes $\Phi_M(F)$ and $\Phi_M(G)$ of the spaces $D_{xy}^\tau, D_{xy}^{\omega_\nu}, J_{xy}^{\omega_\nu}, J_{xy}^\tau$ are respectively the $\Delta\Sigma$-classes $\Phi_{M'_1}(F)$ and $\Phi_{M''_2}(G)$, $\Phi_{M'_1}(F)$ and $\Phi_{M''_2}(G)$, $\Phi_{M_1}(F)$ and $\Phi_{M_2}(G)$, $\Phi_{11'}(F)$ and $\Phi_{M_{12}}(G)$, which may be represented respectively by the operations

$$\bigcup_{\nu_0, \dots, \nu_\alpha, \dots} \Phi_{M'_i} \left\{ \bigcap_{\{\alpha_1, \dots, \alpha_k\} \in I''_\omega} E_{j;\, \nu_{\alpha_1} \dots \nu_{\alpha_k}} \right\} = \Phi_{\tilde{S}', \tilde{M}} \{E_j\}, \qquad E_{j;\, \nu_{\alpha_1} \dots \nu_{\alpha_k}} \in F \subset \mathfrak{P} D^\tau,$$

$$\bigcup_{\nu_0, \dots, \nu_a} \Phi_M \left\{ \bigcup_j \bigcup_{\{\alpha_1, \dots, \alpha_k\} \in I''_\omega} E_j; \nu_{\alpha_1} \dots \nu_{\alpha_k} \right\} = \Phi_{\underline{S}'' \widetilde{M}} \{E_{j\lambda}\}, \qquad E_j; \nu_{\alpha_1} \dots \nu_{\alpha_k} \in G \subset \mathfrak{P} D^\tau,$$

where $\lambda = \lambda(\nu_{\alpha_1}, \dots, \nu_{\alpha_k})$;

$$\bigcup_{\alpha_3,\dots,\nu_\alpha,\dots} \Phi_M \left\{ \bigcup_j \bigcap_\alpha E_j; \nu_0 \dots \nu_\alpha \right\} = \Phi_{\mathfrak{B}\widetilde{M}}\{E_{j\lambda}\}, \qquad E_j; \nu_3 \dots \nu_\alpha \in F \subset \mathfrak{P} D^{\omega_\nu},$$

$$\bigcup_{\alpha_0,\dots,\nu_\alpha,\dots} \Phi_M \left\{ \bigcup_j \bigcup_\alpha E_j; \nu_3 \dots \nu_\alpha \right\} = \Phi_{\mathfrak{B}\widetilde{M}}\{E_{j\lambda}\}, \qquad E_j; \nu_0 \dots \nu_\alpha \in G \subset \mathfrak{P} D^{\omega_\nu},$$

where $\lambda = \lambda(\nu_0, \dots, \nu_\alpha)$;

$$\bigcup_{i_3,\dots,i_\alpha,\dots} \Phi_M \left\{ \bigcup_j \bigcap_\alpha E_j; i_3 \dots i_\alpha \right\} = \Phi_{\mathfrak{A}\widetilde{M}}\{E_{j\lambda}\}, \qquad E_j; i_3 \dots i_\alpha \in F \subset \mathfrak{P} J^{\omega_\nu},$$

$$\bigcup_{i_3,\dots,i_\alpha,\dots} \Phi_M \left\{ \bigcup_j \bigcup_\alpha E_j; i_3 \dots i_\alpha \right\} = \Phi_{\mathfrak{A}\widetilde{M}}\{E_{j\lambda}\}, \qquad E_j; i_3 \dots i_\alpha \in G \subset \mathfrak{P} J^{\omega_\nu},$$

where $\lambda = \lambda(i_0, \dots, i_\alpha)$;

$$\bigcup_{i_0,\dots,i_\alpha,\dots} \Phi_M \left\{ \bigcap_j \bigcup_{\{\alpha_1,\dots,\alpha_k\} \in I'_\omega} E_j; i_{\alpha_1} \dots i_{\alpha_k} \right\} = \Phi_{\underline{S}\widetilde{M}}\{E_{j\lambda}\}, \qquad E_j; i_{\alpha_1} \dots i_{\alpha_k} \in F \subset \mathfrak{P} J^{\tau},$$

$$\bigcup_{i_0,\dots,i_\alpha,\dots} \Phi_M \left\{ \bigcup_j \bigcup_{\{\alpha_1,\dots,\alpha_k\} \in I''_\omega} E_j; i_{\alpha_1} \dots i_{\alpha_k} \right\} = \Phi_{\check{S}\widetilde{M}}\{E_{j\lambda}\}, \qquad E_j; i_{\alpha_1} \dots i_{\alpha_k} \in G \subset \mathfrak{P} J^{\tau},$$

where $\lambda = \lambda(i_{\alpha_1}, \dots, i_{\alpha_k})$.

Corollary 2. The class of projections of sets belonging to the class G of the spaces under study is the class G. The class of projections of sets belonging to the class F in the spaces D^τ, D^{ω_ν}, is contained in the class of (ω_ν) A-sets in the space J^{ω_ν}, and belongs to the class $\Phi_S(F)$ in the space J^τ.

Theorem 2. The bases of the $\Delta\Sigma$-operations $M_1', M_2', M_1'', M_2'', M_1, M_2, M_{11}, M_{12}$ of Corollary 1 can be represented in the form

$$M_1' = \Phi_M\{K_s'\}, \quad M_2' = \Phi_M\{V_s'\}; \quad M_1'' = \Phi_M\{K_s''\}, \quad M_2'' = \Phi_M\{V_s''\};$$

$$M_1 = \Phi_M\{K_s\}, \quad M_2 = \Phi_M\{V_s\}; \quad M_{11} = \Phi_M\{K_s^1\}, \quad M_{12} = \Phi_M\{V_s^1\},$$

where $K'_s, V'_s, K''_s, V''_s, K_s, V_s, K^1_s, V^1_s$ are certain sets of type F_Σ in J^{ω_ν} (independent of M).

The operation $\Phi_\mathfrak{A}$ is incomparable with the operations $\Phi_\mathfrak{B}, \Phi_{S''}$, and at the same time the operation $\Phi_\mathfrak{A}$ is stronger than the operation $\Phi_\mathfrak{B}$ with respect to any class of sets $K \supset \emptyset$, and essentially stronger than $\Phi_{S''}$ and Φ_S with respect to any class of sets invariant with respect to the operation $\bigcap_\tau$. In view of this, the composition $\Phi_{\mathfrak{A}\widetilde{M}}$ is stronger than the composition $\Phi_{\mathfrak{B}\widetilde{M}}$ with respect to the class of sets $K \supset \emptyset$, and stronger than $\Phi_{S''\widetilde{M}}$ and $\Phi_{\breve{S}\widetilde{M}}$ with respect to any class of sets invariant with respect to the operation $\bigcap_\tau$. Consequently, projective operations substantially depend on the topology of the space. Since in the spaces under study the class $F \supset \emptyset$ and is invariant with respect to the operation $\bigcap_\tau$, we have $\Phi_{\mathfrak{A}\widetilde{M}}(F) \supset \Phi_{S''\widetilde{M}}(F)$, $\Phi_{\mathfrak{A}\widetilde{M}}(F) \supset \Phi_{\mathfrak{B}\widetilde{M}}(F)$, $\Phi_{\mathfrak{A}\widetilde{M}}(F) \supset \Phi_{\breve{S}\widetilde{M}}(F)$ for any base M.

The classes of the compositional hierarchies of the space $J^{\omega_\nu}_{xy}$ are constructed as follows. Let $N = \mathfrak{A}$, where $\mathfrak{A}$ is a rigid base of an A-operation with a complete depth chain, $\Phi_M \equiv \bigcup_\tau$. The operation $\Phi_\mathfrak{A}$ and the composition $\Phi_{\mathfrak{A}\widetilde{\mathfrak{A}}}$ are normal, and the operation Φ_M satisfies the condition:

$$1^*.\ \Phi_M \prec \Phi_\mathfrak{A},\ \ \Phi_{Mc} \prec \Phi_\mathfrak{A}.$$

The $\Phi_{\mathfrak{A}M}$-hierarchy of classes of sets generated by the class K_0 of open-closed sets of the space J^{ω_ν} forms the classes of projective sets of this space:

$$P_0 = \Phi^0_{\mathfrak{A}M}(K_0) = \Phi_M(K_0) = \Phi_L(K_0) = G,$$

$$CP_0 = \Phi^{0c}_{\mathfrak{A}M}(K_0) = \Phi^c_M(K_0) = \Phi_{L^c_0}(K_0) = F,$$

$$P_{\alpha+1} = \Phi^{\alpha+1}_{\mathfrak{A}M}(K_0) = \Phi_{\mathfrak{A}\,\breve{L}^c_\alpha}(K_0) = \Phi_{L_{\alpha+1}}(K_0),$$

$$CP_{\alpha+1} = \Phi^{\alpha+1\,c}_{\mathfrak{A}M}(K_0) = \Phi_{\mathfrak{A}\,\breve{L}_\alpha}(K_0) = \Phi_{L^c_{\alpha+1}}(K_0),$$

$$P_\chi = \Phi^\chi_{\mathfrak{A}M}(K_0) = \bigcup_{(\alpha i)\to\chi} \Phi^{\alpha_i}_{\mathfrak{A}M}(K_0) = \Phi_{L_\chi}(K_0),$$

$$CP_\chi = \Phi^{\chi c}_{\mathfrak{A}M}(K_0) = \bigcup_{(\alpha i)\to\chi} \Phi^{\alpha_i c}_{\mathfrak{A}M}(K_0) = \Phi_{L^c_\chi}(K_0),$$

where $\chi < \omega_{\nu+1}$ is a limit transfinite number, $(B_\alpha) = BP_\alpha = P_\alpha \cap CP_\alpha$.

Properties of the classes of projective sets:

1. $P_1 = (A)$, $CP_1 = (CA)$, $B_1 = (A) \cap (CA) = (B)$, where (B) is the class of B-sets of the space J^{ω_ν}.
2. $P_\alpha \subset CP_{\alpha+1}$, $CP_\alpha \subset P_{\alpha+1}$, $P_\alpha \subset P_{\alpha+1}$ for every $\alpha < \omega_{\nu+1}$.
3. The classes P_α for $\alpha < \omega_{\nu+1}$ are invariant with respect to the operation of projection. The classes $P_{\alpha+1}$ and $CP_{\alpha+1}$ are invariant with respect to the operations $\bigcup_\tau$ and $\bigcap_\tau$. The class P_χ is invariant with respect to the operations $\bigcup_\tau$ and intersection of finite families of sets.
4. Under a homeomorphic transformation of the space J^{ω_ν} onto itself, sets of the class $P_\alpha(CP_\alpha)$ pass into sets of the same class $(\alpha < \omega_{\nu+1})$.
5. If $N \in P_{\alpha+1}$, $E_i \in P_{\alpha+1}$ for every $\alpha < \omega_{\nu+1}$, then $E = \Phi_N\{E_i\} \in P_{\alpha+1}$.

In the space J^{ω_ν} the class of projections of B-sets coincides with the class of A-sets of this space, which we shall denote by (A_1). If, in accordance with the projective hierarchy of classes of sets of this space, the projective class (A_α) is defined, then the class of complements to sets of the class (A_α) we shall denote by (CA_α). The class of projections of sets of the class (CA_α) we denote by $(A_{\alpha+1})$. If $\chi < \omega_{\nu+1}$ is a limit transfinite number, then by the class (A_χ) we mean the class of unions of projective sets of classes $< \chi$ of any family of cardinality $\leq \tau$.

Theorem 3. The classes of projective sets (A_α) and (CA_α) for $0 < \alpha < \omega_{\nu+1}$ can be obtained by means of the operations $\Phi_{S^{(i)}_{\alpha+1}}$, for $i = 1, 2, 3, 4$, starting from the classes of sets F and G:

$$(A_\alpha) = \Phi_{S^{(1)}_\alpha}(F) = \Phi_{S^{(2)}_\alpha}(G) = A^\alpha(F);$$

$$(CA_\alpha) = \Phi_{S^{(3)}_\alpha}(F) = \Phi_{S^{(4)}_\alpha}(G) = CA^\alpha(F),$$

where $S^{(1)}_\alpha, S^{(2)}_\alpha \in (CA_{\alpha-1})$, if α is a nonlimit ordinal number, and $S^{(1)}_\alpha, S^{(2)}_\alpha \in (A_\alpha)$, if α is a limit ordinal number; $S^{(3)}_\alpha, S^{(4)}_\alpha \in (CA_\alpha)$ for $\alpha < \omega_{\nu+1}$.

Theorem 4. Each projective operation A^α and CA^α for $0 < \alpha < \omega_{\nu+1}$ can be given by a $\Delta\Sigma$-operation with a rigid base.

Theorem 5. The operations $\Phi_{S^{(i)}_{\alpha+1}}$ for $i = 1, 2$ are normal with respect to any class (A_β) and with respect to classes of sets (CA_β) for $\beta \leq \alpha$, while the operations $\Phi_{S^{(i)}_{\alpha+1}}$ for $i = 3, 4$ are normal with respect to any class (CA_β) and with respect to classes (A_β) for $\beta \leq \alpha$.

Theorem 6. The operations Φ_{L_α} and A^α, $\Phi_{L^c_\alpha}$ and CA^α are equivalent for $\alpha < \omega_{\nu+1}$.

Corollary 1. $P_\alpha = (A_\alpha)$, $CP_\alpha = (CA_\alpha)$ for every $\alpha < \omega_{\nu+1}$.

Corollary 2. In the space J^{ω_ν}, the class of C-sets belongs to the class $(\overline{B}_2)$.

Corollary 3. Complete bases of all $(\omega_\nu)R^\alpha$-operations belong to the class $(\overline{B}_2)$. The class of $(\omega_\nu)R$-sets of the space J^{ω_ν} is contained in the class (B_2).

Corollary 4. If the class $P_{\alpha+1}$ is invariant with respect to all operations with bases of the family $\mathfrak{M} = (M_i) \cup (M_{(i\alpha)_\gamma})$, where $i \in I$, $(i\alpha)_\gamma$ is the totality of all concordant sequences of limit type $\gamma \leqslant \omega_\nu$, then it is also invariant with respect to the operation $(\omega_\nu)T$.

It follows from this that the operation of type $(\omega_\nu)T_{\{L_{\alpha+1}\}}$ is equivalent to the operation $\Phi_{L_{\alpha+1}}$ with respect to any class of projective sets for any $\alpha < \omega_{\nu+1}$; the operation of type $(\omega_\nu)P_{\mathfrak{A}\check{\mathfrak{M}}^c}$, where $\Phi_M \equiv \bigcup_\tau$, is equivalent to the operation $\Phi_{\check{\mathfrak{A}}L_1^c} \equiv \Phi_{L_2}$; the operation $(\omega_\nu)P_{\check{\mathfrak{A}}L_{\alpha+1}^c}$ is equivalent to the operation $\Phi_{L_{\alpha+2}}$ with respect to any class of projective sets for $0 < \alpha < \omega_{\nu+1}$.

When the generalized continuum axiom is added, for the class of sets P_2 the separation laws that hold are the same as for the second class of projective sets of the space J.

Let the base be $N = S$, $\Phi_M \equiv \bigcup_\tau$. The operation Φ_S and the composition $\Phi_{\check{S}}\check{S}$ are normal (the latter with respect to the class $K \supset J^\tau$), and the operation Φ_M satisfies condition 1^*. The Φ_{SM}-hierarchy of classes of sets generated by the class K_0 of open-closed sets of the space J^τ forms classes of projective sets of this space, which we shall denote in the same way as for the space J^{ω_ν}. The class $P_0 = G$, $CP_0 = F$, $(B_0) = K_0$, $P_1 \subset (A)$, $CP_1 \subset (CA)$, $(B_1) \subset (B)$. The class of projections of B-sets of the space J^τ coincides with the class of sets P_1. The class of $(\omega_\nu)A$-sets of this space contains, as a proper part, the class of projections of B-sets of the given space.

Let the base be $N = S''$; the operation Φ_M is equivalent to the operation $\Phi_{M_1}\{E_{ij}\} = \bigcup_i \bigcap_j E_{ij}$. The operation $\Phi_{S''}$ is nonnormal, but the composition $\Phi_{\check{S}''S''}$ is normal with respect to the class $K \supset D^\tau$. The operation Φ_M does not satisfy condition 1^*. The $\Phi_{S''\check{M}}$-hierarchy of classes of sets, generated by the class K_0 of open-closed sets of the space D^τ, forms a hierarchy of projective classes of the space D^τ. In this case the class $P_0 = G^2$, $CP_0 = F^2$, $(B_0) = B^2$. The classes of projective sets are monotone. For even α the class P_α is invariant with respect to the operation $\bigcup_\tau$, and the class $P_{\alpha+1}$ with respect to the operation $\bigcap_\tau$. The class $P_1 \subset (A)$ in the space D^τ.

Let the base be $N = \mathfrak{B}$; Φ_M is equivalent to the operation $\Phi_{M_1}\{E_{ij}\} = \bigcup_i \bigcap_i E_{ij}$. The operation $\Phi_{\mathfrak{B}}$ is nonnormal, but the composition $\Phi_{\check{\mathfrak{B}}\mathfrak{B}}$ is normal with respect to the class of sets $K \supset D^{\omega_\nu}$. The operation Φ_M does not satisfy condition 1^*. The $\Phi_{\mathfrak{B}\check{M}}$-hierarchy of classes of sets forms a projective hierarchy of the space D^{ω_ν}. The properties of the classes of sets P_α of the space D^{ω_ν} coincide with the properties of the corresponding classes of the space D^τ. $P_1 \subset (A)$ in the space D^{ω_ν}.

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Note: Figure translations are in progress. See original paper for figures.