FACTORIZATION OF FINITE GROUPS

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Abstract

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MATHEMATICS

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§ 1. A subgroup H is called an **addition to a normal subgroup** N of a finite group G if $HN = G$ and $MN \neq G$ for every nontrivial subgroup M of H (in the particular case when $H \cap N = 1$, the subgroup H is called a **complement**). Apparently, P. Hall was the first to note that an addition has the following properties: the intersection $H \cap N$ is nilpotent and the sets of prime divisors of the orders of the groups H and G/N coincide (see ([1])).

These properties may be obtained as a consequence of the fact that the intersection $H \cap N$, as follows from the definition itself, must lie in the Frattini subgroup of the group H. However, despite its simplicity, the noted result has played a significant role in the works of various authors, serving as a starting point in finding subgroups and methods of factorization of groups.

The next important step in the study of additions was made by Huppert ([2]), who found that if an addition H to a normal subgroup N of a finite group G is a Sylow p-subgroup in G and $p > 2$, then H has a normal complement in G. The restriction $p > 2$ arose because Huppert' s proof was based on Thompson' s theorem ([3]) on the existence of normal p-complements for $p > 2$; however, Thompson later showed that Huppert' s result remains true also for $p = 2$. A simple proof of these results was given by Tate by cohomological methods ([4]), and a purely group-theoretic proof was given by Rokette ([5]). At the same time it was established that, instead of Sylow subgroups, one may with the same success consider additions that are Hall subgroups.

Continuing these investigations, we have obtained the following two theorems.

Theorem 1. *Let K be the subgroup of a finite group G generated by all the Π'-elements from G, and let H be an arbitrary addition to K. Then $H \cap K$ contains no Sylow subgroup (distinct from the identity) of the group K.*

Here and below Π denotes some set of prime numbers; Π' is the complement to Π in the set of all prime numbers.

We note that Rokette' s theorem ([5]) follows from Theorem 1.

Theorem 2. *Let K be some normal subgroup of a finite group G, and let H be an addition to K, and suppose that for every prime number p dividing the order of $H \cap K$, a Sylow p-subgroup P of the group G satisfies the condition: the intersection $P \cap K$ is an elementary abelian subgroup of the center of P. Then H is a complement to some normal subgroup of the group G contained in K.*

According to Schur' s theorem, if a subgroup K, generated by the Π'-elements of a finite group G, is a Π'-group, then K has a complement in G. From Theorem 2 the following generalization of this theorem of Schur follows.

Theorem 3. *Let K be the subgroup of a finite group G generated by all Π'-elements from G. If the order of K is not divisible by p^2 for any $p \in \Pi$, then K has at least one complement in G.*

Under the conditions of Theorem 3, the Sylow p-subgroups of K for $p \in \Pi$ will be

either trivial or cyclic of prime order. It is natural to ask to what extent this condition on Sylow subgroups can be weakened. So far such a weakening has been achieved only for one Sylow subgroup. Namely, Theorem 3 remains valid if in it the subgroup K satisfies the following weaker condition: K contains a Sylow p-subgroup, $p \in \Pi$, which is abelian for $p = 2$, modular for $p > 2$, and the order of K is not divisible by q^2 for any $q \in \Pi,\ q \neq p$.

§ 2. Let us dwell on some details of the proof. The first two theorems are proved by studying the following situation.

Situation A. Let H, H_1, H_2, H_3, K be certain subgroups of a finite group G. Let $p_1, \dots, p_t$ be all the distinct prime divisors of the order of H_3. For each $i = 1, \dots, t$ choose a Sylow p_i-subgroup P_i of H and a Sylow p_i-subgroup T_i of G such that $T_i \supseteq P_i$. Suppose the following conditions are satisfied:

a) $G = HK,\ H \cap K = H_1,\ K$ is normal in G;

b) H_2 is a Hall subgroup of the group H_1 and is normal in H;

c) $H_3 H_2 = H_1,\ H_3 \cap H_2 = 1$;

d) H_1/H_2 is contained in the Frattini subgroup of the group H/H_2;

e) $T_i = P_{iL}i,\ P_i \cap L_i = 1,\ L_i \subseteq K,\ L_i$ is normal in T_i $(i = 1, \dots, t)$.

Proposition B. Suppose Situation A holds and suppose that H contains a normal subgroup R such that $H_1 \supseteq R \supseteq H_2$ and H_1/R is abelian. Then there exists a homomorphism φ of the group G onto H/R whose kernel is contained in K.

Proposition C. Suppose Situation A holds, except that in place of condition e) the following condition is satisfied:

e$'$) if $P_i \neq T_i$, then $T_i \cap K$ is an elementary abelian subgroup contained in the center of the subgroup $T_i,\ i = 1, \dots, t$.

Then G has such a normal subgroup N that $G = HN,\ N \subseteq K,\ N \cap H = H_2$.

This proposition contains Theorems 1 and 2.

Proposition D. Let a finite group G have a normal Sylow p-subgroup P, which is modular, and for $p = 2$ abelian. Let Q be a complement to P, and let R be the subgroup generated by all p'-elements of G. Then

$$G = RN_G(Q), \qquad R \cap N_G(Q) = Q.$$

For the case when P is abelian, this fact is known (see (6)). The generalization to the modular case became possible thanks to Huppert' s remarkable result (7) on automorphisms of regular p-groups (we recall that a group is called modular if the lattice of all its subgroups is modular).

Proposition E. Let a finite group G have a p-solvable normal subgroup K generated by its p'-elements. Let a Sylow p-subgroup P of K be abelian for $p = 2$, modular for $p > 2$. Then P has in G at least one complement, and all such complements in G are conjugate.

In the case when P is abelian and normal in G, this proposition becomes the theorem of H. Higman (6).

Proposition F. Let G have a normal subgroup K, generated by its p'-elements (p a fixed prime), and let a Sylow p-subgroup P of K be abelian for $p = 2$, modular for $p > 2$. Then there exists a complement H to the subgroup K in the group G such that $H \cap P = 1$.

§ 3. In a number of papers (see (8)) S. A. Chunikhin consistently studied the possibility of factorizing an arbitrary finite group by means of the indices of its chief series. The works of M. I. Kargapolov (9–12) also belong to this direction. The latest results in this direction were two theorems (12, 13). The theorem of S. A. Chunikhin (13) corre-

assigned to each partition of the set K of indices of the chief series into subsets K_i a certain factorization of the group by pairwise permutable subgroups. The factorization defined by the theorem of M. I. Kargapolov ([12]) corresponded to that special case of S. A. Chunikhin' s theorem in which the sets K_i are one-element; however, the advantage of theorem ([12]) consisted in the detailed information about the connection of the factors of the factorization found with the members of the chief series.

To find a similar connection for the factors of S. A. Chunikhin' s theorem (and thereby to establish the connection between theorems ([12]) and ([13]))—this problem was posed by S. A. Chunikhin in his monograph ([8]), p. 79. We give the solution of this problem found by us.

Take an arbitrary series of a finite group G composed of its normal subgroups (an invariant series):

$$1 = G_0 \leq G_1 \leq \cdots \leq G_t = G, \quad t \geq 1. \tag{R}$$

A collection $R_1, R_2, \dots, R_t$ of pairwise permutable subgroups will be called a **decomposition** of the group G, **corresponding to the series** (R), if for every $i = 1, 2, \dots, t$ the following conditions are satisfied:

a) $R_{iG_{i-1}} = G_i$;

b) $R_i \cap G_{i-1}$ is a nilpotent Π_i-group, where Π_i is the set of all distinct prime divisors of the index $(G_i : G_{i-1})$.

Theorem 4. *To every invariant series of a finite group G there corresponds at least one decomposition of the group G.*

We note that a decomposition corresponding to a chief series of the group G will at the same time be a chief decomposition in the sense of M. I. Kargapolov ([12]); the converse is, obviously, false. Thus, Theorem 4 improves the factorization of theorem ([12]).

Let now (R) be a chief series of the finite group G, and let $\{R_i\}$ be a decomposition existing by Theorem 4. From the sequence K of indices of the series (R) choose such subsequences $K_1, \dots, K_\mu$ whose union coincides with K. Denote by k_i the product of all elements of K_i. For each $i = 1, \dots, \mu$ construct a subgroup $\mathfrak{R}_i$, setting it equal to the product of all those members of the decomposition $\{R_i\}$ which correspond to indices from K_i.

From the definition of a decomposition it follows that $R_i \cap R_j$, for $i > j$, is contained in G_{i-1}; therefore the order of the subgroup $R_{iR}j$ is divided by the product of the corresponding indices of the series (R). A similar fact is also true for the product of several members of the decomposition. We thus obtain the following factorization:

$$G = \mathfrak{R}_1 \mathfrak{R}_2 \cdots \mathfrak{R}_\mu, \quad \mathfrak{R}_i \mathfrak{R}_j = \mathfrak{R}_j \mathfrak{R}_i \quad \text{for } 1 \leq i, j \leq \mu, \quad |\mathfrak{R}_i| = k_i c_i,$$

where the prime divisors of the number c_i are contained among the prime divisors of the number k_i. This is precisely the factorization found by S. A. Chunikhin ([13]); the connection of the factors $\mathfrak{R}_i$ with the members of the series (R) is effected through the members of the decomposition $\{R_i\}$.

Theorem 5. *Let $\{R_i\}$ be a decomposition of a finite group G corresponding to its chief series (R). Let I be a collection of (possibly not all) prime indices of the series (R). Then the product of all those members of the decomposition $\{R_i\}$ which correspond to indices from I is soluble and contains a supersoluble subgroup of order h, where h is the product of all elements of I.*

§ 4. Let J be the collection of all those indices of some chief series of a finite group G which are prime numbers. If J is nonempty, then the product of all elements of J is called, according to S. A. Chunikhin ([14]), the measure of supersolubility of the group G. If J is empty, then the measure of supersolubility of the group G, by definition, is equal to 1.

From S. A. Chunikhin's theorem ([14]) on indexials it follows that if h is a divisor of the measure of supersolubility of a finite group G, then G has a soluble subgroup of order divisible by h. It is also known that finite supersoluble groups possess subgroups of every possi-

possible order (15). From Theorem 5 there follows the following exact result.

Theorem 6. Let c be the measure of supersolvability of a finite group G. Then in G there exists a supersolvable subgroup of order h for every divisor h of the number c. In particular, G contains at least one supersolvable subgroup of order c.

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Note: Figure translations are in progress. See original paper for figures.