FIXED POINTS OF NONCOMPACT MAPPINGS

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MATHEMATICS

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FIXED POINTS OF NONCOMPACT MAPPINGS

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1. In the present note we give a construction which originally arose as a new simple proof of the Schauder–Tikhonov principle. At the same time this construction made it possible to obtain a number of new existence theorems for fixed points (Theorems 1-6).

Let B be a Banach space and B^* its conjugate space. Denote by $D(B)$ the unit ball ($D(B) = \{x \in B : \|x\| \leq 1\}$). Let $K \subset D(B)$ be a convex closed set, and let a set $F \subset B^*$ separate the points of K (i.e., for $x_1, x_2 \in K$ and $x_1 \neq x_2$ there exists $f \in F$ such that $f(x_1) \neq f(x_2)$). Suppose that K is compact in the weak topology induced by the functionals from the linear set $F \subset B^*$ (the so-called $\sigma(F)$-topology; we shall sometimes simply write weak topology). Here $\widetilde{F} \subset F$. We shall be interested in conditions on a mapping $A : K \to K$ under which a fixed point exists.

2. **Lemma 1.** Let K_n be a convex compactum in E_n ($\dim E_n = n$); let A be a multivalued mapping of K_n, namely $Ax = C(x)$, where $C(x)$ is a convex closed set in K_n. Suppose the mapping has a closed graph (i.e., the closed set $\Gamma_A = \{(x, Ax)\}_{x \in K_n} \subset K_n \times K_n$). Then there exists $x \in K_n$ such that $x \in C(x)$.

The proof is carried out by reduction to the usual Brouwer principle.

3. **The basic construction.** Let $A : K \to K$ and suppose the conditions of item 1 are satisfied. For $F_n = \{f_i\}_{i=1}^n \subset F$ consider

$$P(F_n)x = (f_1(x), \dots, f_n(x)) = y^{(n)} \in E_n.$$

$P(F_n)$ is a mapping of K into n-dimensional space, and

$$P(F_n)K = K(F_n)$$

is a convex closed set in the space

$$B\Big/\left\{\bigcap_{i=1}^{n}(x : f_i(x) = 0)\right\}.$$

For $y \in K(F_n)$ denote the complete inverse image in K by

$$\mathfrak{X}_y = P^{-1}(F_n)y = \{x : P(F_n)x = y \text{ and } x \in K\}.$$

Note that if Y is a closed set in $K(F_n)$, then

$$\mathfrak{X}_Y = P^{-1}(F_n)Y$$

is a closed set in K in the $\sigma(F)$-topology.

We now define a multivalued operator $\bar{A}(F_n)$, acting in $K(F_n)$, by the formula

$$\bar{A}(F_n)y_0 = \{y_\alpha \in K(F_n) : \exists x_\alpha \in \mathfrak{X}_{y_0} \text{ and } P(F_n)Ax_\alpha = y_\alpha\}.$$

In other words,

$$\bar{A}(F_n)y_0 = P(F_n)AP^{-1}(F_n)y_0.$$

From the operator $\bar{A}(F_n)$ we construct the operator $A(F_n)$:

$$A(F_n)y_0 = \operatorname{conv}(\bar{A}(F_n)y_0),$$

where $\operatorname{conv} M$ denotes the convex closed hull of the set M.

Requirement on the operator A:

(I). For every F'_m there exists $F_n \supset F'_m$ such that $A(F_n)$ is continuous* at every interior point of $K(F_n)$.

* A multivalued mapping $A : K \to K$ is continuous if, for every $x_0 \in K$ and $\varepsilon > 0$, there exists $\delta > 0$ such that from $\rho(x, x_0) < \delta$ it follows that

$$\rho(Ax, Ax_0) = \max\left\{\sup_{y \in Ax} \rho(y, Ax_0);\ \sup_{y \in Ax_0} \rho(y, Ax)\right\} < \varepsilon.$$

Lemma 2. If the operator A is uniformly continuous on K, then (I) is satisfied.

The requirement of continuity of $A(F_n)$ only at the interior points of $K(F_n)$ is caused by the arbitrariness of K. In the case when A is weakly continuous (the Schauder-Tikhonov case), or $K = D(B)$ and is locally uniformly convex (see (3), p. 188), all $A(F_n)$ are continuous on all of $K(F_n)$. Below, for simplicity of exposition, precisely this is assumed; however, Proposition 1 is formulated in full generality.

By Lemma 1, from the continuity of $A(F_n)$ there follows the existence of $y_0 \in K(F_n)$ such that $y_0 = A(F_n)y_0$. Denote the set of such y by $N(A; F_n)$, and its complete inverse image in K, $P^{-1}(F_n)N = \mathfrak{N}(A; F_n)$. From the preceding it is clear that $\mathfrak{N}(A; F_n)$ is a closed set in K in the $\sigma(F)$-topology (since $N(A; F_n)$ is closed in $K(F_n)$) and, as is easy to show, is a centered family with respect to $\{F_n\}$. Hence:

Proposition 1. If condition (I) on A is satisfied, then there exists

$$\emptyset \neq \bigcap_{F_n \subset F} \mathfrak{N}(A; F_n) = \mathfrak{N}(A) \subset K.$$

4. We shall now be interested in conditions on A under which $\mathfrak{N}(A)$ consists of fixed points of the mapping A. Let $x_0 \in \mathfrak{N}(A)$, $x_1 = Ax_0$. Suppose that $x_1 \neq x_0$. By the construction of $\mathfrak{N}(A)$, this means that for every F_n the set*

$$H_{F_n}(x_0) \equiv H_{F_n}(\bar{a}) = \{x \in K : \ f_i(x) = f_i(x_0) \text{ for } f_i \in F_n\},$$

where $\bar{a} = (f_1(x_0), \dots, f_n(x_0))$, is carried by the operator A into a set whose closed convex hull contains x_0. Thus:

Proposition 2. If for the mapping $A : K \to K$ there is satisfied

$$\text{(II) for every } x_0 \in K, \quad Ax_0 \neq x_0, \quad \exists F_n = \{f_i\}_{i=1}^n$$

such that

$$x_0 \notin \operatorname{conv} AH_{F_n}(x_0),$$

then $\mathfrak{N}(A)$ coincides with the set of fixed points of the mapping A.

Remark 1. The Schauder–Tikhonov principle itself has in fact already been proved, since if A is continuous in the $\sigma(F)$-topology, then, as shown above, $\mathfrak{N}(A) \neq \emptyset$, and (II) is easily verified.

Indeed, if $Ax_0 = x_1 \neq x_0$, then there exists a separating functional $f_0 \in F$, i.e. $f_0(x_0) \neq f_0(x_1)$. This means that x_1 has a neighborhood in the $\sigma(F)$-topology not containing x_0. Such a neighborhood is

$$H_{f_0,\varepsilon_0}(x_1) = \{x \in K : \ |f_0(x) - f_0(x_1)| \leq \varepsilon_0\}.$$

The complete inverse image of this neighborhood is a neighborhood of x_0 by the continuity of A in the $\sigma(F)$-topology. Hence there exist $F_n = \{f_i\}_{i=1}^n$ and $\varepsilon > 0$ such that the neighborhood $V(x_0)$ of the point x_0

$$H_{F_n,\varepsilon}(x_0) = \{x \in K : \ |f_i(x) - f_i(x_0)| < \varepsilon \text{ for } i = 1, \dots, n\} \subset A^{-1} H_{f_0,\varepsilon_0}(x_1).$$

It follows that (II) is satisfied, since the image $H_{F_n,\varepsilon}$ is contained in the closed convex set $H_{f_0,\varepsilon_0}(x_1)$, which does not contain x_0. Let us also note that, in a completely analogous way, one obtains a generalization of the Schauder–Tikhonov principle for multivalued mappings (Kakutani theorem, see (1, 2))—an infinite-dimensional analogue of Lemma 1.

In what follows we present various variants of requirements on A and K under which (II) is satisfied and, consequently, the mapping A has a fixed point. Uniform continuity of A on K (abbreviated u.c. A) will be assumed in the sequel.

5. First we give a proposition on superposition often used in examples.

* We introduce two notations for this set.

Proposition 3. If the mapping A_0 satisfies requirement (II), and A_1 and A_2 are continuous in the $\sigma(F)$-topology, then the mapping $A_2A_0A_1$ satisfies requirement (II).

Theorem 1. Let the operator A_0 satisfy the following condition: there exists a set $F \subset F$ separating points of K, such that for every $f \subset F$ and every a there is a b such that $A_0H_f(a) \subset H_f(b)$. Let A_1 and A_2 be continuous in the $\sigma(F)$-topology and let $A_1A_0A_2$ be a r.n. operator $K \to K$. Then the mapping $A = A_1A_0A_2$ has a fixed point, and $\mathfrak{N}(A)$ is the set of all fixed points of A.*

Example-theorem 1a. Let $B(G)$ be the Banach space of functions defined on $G \subset E_n$, and suppose that for any $\alpha \in G$ the functionals $f_\alpha\colon f_\alpha(x(t)) = x(\alpha)$, where $t \in G$ and $x(t) \in B(G)$, are continuous. Denote $\{f_\alpha\}_{\alpha\in G} = \widetilde{F}$. Let $K \subset D(B(G))$ and let it be weakly compact in the $\sigma(F)$-topology, where $\widetilde{F} \subset F$. Then, for any function $\varphi(t; u)$ continuous in u, the Hammerstein operator

$$Ax = \int_G K(s,t)\varphi(t; x(t))\,dt$$

has a fixed point if and only if $A\colon K \to K$ and is uniformly continuous on K.

Theorem 1 was given by us for greater transparency. The more general Theorem 2 below also makes it possible to analyze the case of the Urysohn operator:

$$Ax = \int_G K(s,t; x(t))\,dt.$$

Theorem 2. Suppose that for the mapping $A_0\colon K \to K$ there exists a set $F \subset F$ separating points of K, such that for every $x_0 \in K$, every $F_n \subset F$, and $\varepsilon > 0$ there are $F_m = \{f_i\}_{i=1}^m \supset F_n$ and $b = (b_1, \dots, b_m)$ such that

$$A_0H_{F_m}(x_0) \subset H_{F_m,\varepsilon}(b).$$

Then for any mappings A_i $(i = 1, 2)$ continuous in the weak topology and such that $A_2A_0A_1\colon K \to K$ and r.n. on K, the set $\mathfrak{N}(A_2A_0A_1) \neq \emptyset$ and coincides with the set of fixed points of the mapping $A_2A_0A_1$.*

Theorem 3. Let the linear span F be dense in B^* (this is always fulfilled, for example, for reflexive spaces). Consider r.n. mappings $A\colon K \to K$, where

$$A = \sum_{i=1}^{n} A_2^{(i)} A_0^{(i)} A_1^{(i)}.$$

If each summand is continuous in B and satisfies the conditions of Theorem 2 (but does not necessarily carry $K \to K$), then $\mathfrak{N}(A) \neq \emptyset$ and coincides with the set of fixed points of the mapping A.

6. Asymptotically weakly continuous and asymptotically affine mappings. We shall call a mapping A **asymptotically weakly continuous** if, for every $x \in K$ and $\varepsilon > 0$, there is an $E^{N(x,\varepsilon)}$ ($\operatorname{codim} E^N = N$) such that the operator A, considered on $(x + E^N) \cap K$, is an ε-perturbation of a weakly continuous one, i.e., for

$$M = A[(x + E^N) \cap K]$$

and every $U(Ax) \in \sigma$ there is a $V(x) \in \sigma$ such that**

$$AV(x) \subset M_\varepsilon \cap U(Ax).$$

We shall call a mapping $A\colon K \to K$ **asymptotically affine** if, for every $x \in K$ and $\varepsilon > 0$, there is an $E^{N(x,\varepsilon)}$ ($\operatorname{codim} E^N = N$) such that

$$\left\| A\left(\frac{x_1 + x_2}{2}\right) - \frac{Ax_1 + Ax_2}{2} \right\| < \varepsilon \quad \text{for } x_i \in (x + E^N) \cap K \ (i = 1, 2).$$

Theorem 4. For the classes of mappings indicated in this section, requirement (II) is fulfilled and, consequently, under the condition of uniform continuity of these mappings they possess fixed points.

* A more general assertion is true, with A_0, A_1, and A_2 acting in different vector spaces.

** M_ε denotes the ε-enlargement of the set M.

7. **Theorem 5.** If a u.c. mapping $A : K \to K$ satisfies the condition: for any $x_0 \in K$, $f \in F$, and $\varepsilon > 0$ there is an $F_n \subset B^*$, $f \in F_n$, such that $d(AH_{F_n}(x_0)) < \varepsilon$, where $d(M)$ is the diameter of the set M, then $\emptyset \neq \mathfrak{N}(A)$ coincides with the fixed points of the mapping A.

8. **On one class $\mathfrak{A}$ of mappings** $K \to K$**.** Instead of the sets $\mathfrak{N}(A; F_n)$, introduced in Sec. 3 of the present paper, consider their closed ε-neighborhoods $\mathfrak{N}_\varepsilon(A; F_n)$. Denote

$$\mathfrak{N}_\varepsilon(A) = \bigcap_{(F_n) \subset F} \mathfrak{N}_\varepsilon(A; F_n)$$

and

$$\rho_A(M) = \sup\{\|Ax - x\| \text{ for } x \in M\}.$$

Definition of the class $\mathfrak{A}$. $A \in \mathfrak{A}$ if and only if $\rho(\mathfrak{N}_\varepsilon(A)) \to 0$ as $\varepsilon \to 0$ and A is uniformly continuous.

Obviously, if $A \in \mathfrak{A}$, then $\mathfrak{N}(A)$ coincides with the set of fixed points of the mapping A. Note that in all the examples of mappings considered above (Theorems 1-5), the operators belonged to the class $\mathfrak{A}$. Moreover:

Theorem 6. Any finite superposition of the operators indicated in Theorems 1-5 is also an operator from $\mathfrak{A}$.

However, the introduction of the class $\mathfrak{A}$ is prompted by the following theorem.

Theorem 7. The class $\mathfrak{A}$ is closed in the uniform operator topology.

Combining this theorem with Theorem 3 and with Example-Theorem 1a, one can obtain fixed-point theorems for a broad class of Urysohn operators.

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Note: Figure translations are in progress. See original paper for figures.