COMPLEXES OF \(k\)-DIMENSIONAL PLANES IN THE SPACE \(C^n\) AND THE PLANCHEREL FORMULA FOR THE GROUP \(GL(n,C)\)

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Abstract

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COMPLEXES OF k-DIMENSIONAL PLANES IN THE SPACE C^n AND THE PLANCHEREL FORMULA FOR THE GROUP $GL(n, C)$

1. Let $\bar{H} = H_{n,k}$ be the set of all k-dimensional planes in C^n, endowed in the natural way with the structure of a complex analytic manifold. By a **complex** K of k-dimensional planes in C^n we shall mean any closed irreducible algebraic submanifold in H, whose (complex) dimension is equal to n, and such that through every point $x \in C^n$ (with the possible exception of a manifold of points of lower dimension) there passes at least one plane $h \in K$.*

If a complex K of k-dimensional planes in C^n is given, then the following problem of integral geometry may be formulated for it. To each function $f(x) \in S$ (S is the space of infinitely differentiable and rapidly decreasing functions on C^n) assign its integrals $\varphi(h)$ over the planes $h \in K$. It is required to obtain an inversion formula, i.e., to recover the original function $f(x)$ from the function $\varphi(h)$.

In the paper a class of **admissible** complexes is defined, for which the inversion formula follows directly from the results of ([1],[2]). In Sec. 2, for the case $k = 1$ (i.e., for complexes of lines), a simple geometric description of admissible complexes in general position is obtained.

In Sec. 3 an example is considered of a complex of $n(n-1)/2$-dimensional planes in C^{n^2}, arising in a natural way in the representation theory of the group $GL(n, C)$. It is proved that this complex is admissible, and an inversion formula is obtained for it. This inversion formula was previously proved by other methods in ([3],[4]). The method by which it is obtained in the present paper is, in the authors' opinion, the simplest and most natural.

We shall briefly present the results of the paper ([2]). We shall specify any k-dimensional plane h in C^n by the equation $x = \alpha t + \beta$, where α is a matrix whose columns are k linearly independent vectors $\alpha_1, \dots, \alpha_k \in C^n$, $\beta \in C^n$, and t ranges over C^k. Next fix an infinitely differentiable function $g(\alpha)$, not vanishing anywhere and satisfying, for any nondegenerate linear transformation μ in C^k, the following condition: $g(\alpha\mu) = g(\alpha)|\det \mu|^2$. To each function $f(x) \in S$ we associate a function $\varphi(h)$ on $H_{n,k}$ by the formula

$$\varphi(h) = g(\alpha) \int f(\alpha t + \beta)\, dt\, d\bar{t}. \tag{1}$$

(It is easy to verify that the integral does not depend on the parametric specification of the plane h.)

In the paper ([2]), for each point $\beta \in C^n$ a linear mapping was defined

$$\varkappa_\beta : \Phi_H \to \Phi_{G_\beta}^{(k,k)}$$

from the space Φ_H of all infinitely differentiable functions on $H_{n,k}$ into the space $\Phi_{G_\beta}^{(k,k)}$ of differential forms of type (k, k) defined on the submanifold G_β of k-dimensional planes passing through the point β. The following inversion formula for the integral was established

* For simplicity, manifolds without singularities are considered, although all results remain valid also for man

of the transform (1):

$$\int_\gamma \chi_\beta \varphi = c_\gamma f(\beta), \tag{2}$$

where the integral is taken over an arbitrary cycle $\gamma \subset G_\beta$ of real dimension $2k$, and the factor c_γ depends only on the homology class to which the cycle γ belongs.

Definition. Let K be a complex analytic submanifold of $H_{n,k}$ of dimension $\geqslant n$ (in particular, a complex), and let

$$i^* : \Phi_H \to \Phi_K, \qquad \Phi_{G_\beta}^{(k,k)} \to \Phi_{G_\beta \cap K}^{(k,k)}$$

be the natural mappings of functions and differential forms. We shall call the submanifold K **admissible** if, for almost every $\beta \in C^n$, there exists a linear mapping

$$\chi'_\beta : \Phi_K \to \Phi_{G_\beta \cap K}^{(k,k)}$$

such that the diagram

$$\begin{array}{ccc} \Phi_H & \xrightarrow{\chi_\beta} & \Phi_{G_\beta}^{(k,k)} \\ i^* \downarrow & & \downarrow i^* \\ \Phi_K & \xrightarrow{\chi'_\beta} & \Phi_{G_\beta \cap K}^{(k,k)} \end{array}$$

is commutative. (It is clear that in this case χ'_β is uniquely determined and is, just like χ_β, a differential operator.) If K is an admissible submanifold, then, by virtue of (2), the following inversion formula for (1), in which h ranges over K, is valid:

$$\int_\gamma \chi'_\beta \varphi = c_\gamma f(\beta),$$

where γ is an arbitrary cycle of real dimension $2k$ in $G_\beta \cap K$. In particular, if K is a complex, then for points $\beta \in C^n$ in general position $\dim(G_\beta \cap K) = 2k$, and therefore one may take $\gamma = G_\beta \cap K$. (Of course, this formula determines $f(\beta)$ only if $c_\gamma \neq 0$.)

2. Admissible complexes of lines in C^n

Theorem 1. *A complex of lines K is admissible if and only if, for almost every line $h \in K$, the following condition holds: if one removes from the line h some finite number of points and takes the submanifold of lines in K that meet h at the remaining points, then the closure in K of this submanifold is nonsingular at the point $h \in K$.*

In what follows we shall assume that C^n is embedded in the projective space CP^n, and thereby each line h is completed by a point at infinity. Let K be a complex of lines, $h \in K$, and $x \in C^n$ a point on the line h. Consider the tangent space τ_H at h to the manifold H of all lines, and take in it two subspaces—the tangent space τ_K to the complex K and the tangent space τ_{G_x} to the manifold G_x of all lines passing through x. Since $\dim \tau_H = 2n - 2$, $\dim \tau_K = n$, and $\dim \tau_{G_x} = n - 1$, in general position we have:

$$\dim(\tau_K \cap \tau_{G_x}) = 1.$$

We shall call a point x of the line $h \in K$ a **critical point** if

$$\dim(\tau_K \cap \tau_{G_x}) > 1.$$

Let us find all critical points of the line $h \in K$. Suppose, for definiteness, that h is not parallel to the hyperplane $x^n = 0$. Then every line close to h can be given by a system of equations

$$x^i = a^i x^n + \beta^i x^0, \quad i = 1, \dots, n-1,$$

where $(x^0, \dots, x^n)$ are homogeneous coordinates in CP^n. We take the $2n - 2$ complex numbers a^i, β^i as local coordinates on the manifold of all lines in a neighborhood of h, and let the complex K be defined in a neighborhood of h by the equations

$$u^1(a, \beta) = 0, \dots, u^{n-2}(a, \beta) = 0.$$

Then, in order that the point $x = (x^0, \dots, x^n) \in h$ be critical, it is necessary and sufficient that the rank of the matrix

$$\left\| x^0 \partial u^i / \partial a^j |_h - x^n \partial u^i / \partial \beta^j |_h \right\|_{i=1,\dots,n-2;\; j=1,\dots,n-1}$$

be less than $n-2$. Let $\Delta_i(x^0, x^n)$ be the minors of order $n-2$ of this matrix. It can be shown that, in the case of an admissible complex, they dis-

differ by factors that do not depend on x. Thus, if $\Delta_i \not\equiv 0$ for at least one i, then (since Δ_i is a homogeneous polynomial of degree $n-2$ in x^0, x^n) on the projective line h there are, counting multiplicities, $n-2$ critical points. If, however, $\Delta_i \equiv 0$ for all i, then all points of the line h are critical (such lines form in K a submanifold of lower dimension).

Theorem 2. *The critical points of the lines of an admissible complex form in* $\mathbf{CP}^n$ *a manifold of dimension less than* n.

We shall call an admissible complex of lines K a **complex in general position** if on each line $h \in K$ (with the possible exception of a submanifold of lines of lower dimension) there are exactly $n-2$ pairwise distinct critical points. We describe the local structure of such complexes.

Theorem 3. *Suppose that on a line* h_0 *of an admissible complex* K *there are exactly* $n-2$ *pairwise distinct critical points* $x_1, \dots, x_{n-2}$. *Then the critical points of the lines* $h \in K$ *close to* h_0 *describe* $n-2$ *local algebraic surfaces* $M_1, \dots, M_{n-2}$, *each of which has dimension* $n-1$ *or* $n-2$; *moreover, if* $\dim M_i = n-1$, *then the lines* h *are tangent to* M_i.

Assume further that the points $x_i \in M_i$ *are nonsingular, and let* P_i *be the hyperplane in* $\mathbf{CP}^n$ *spanned by the tangent plane to* M_i *at* x_i *and the line* h_0 $(i = 1, \dots, n-2)$. *Then, if the hyperplanes* P_i *are in general position, the set of all lines in* $\mathbf{C}^n$ *tangent to each local surface* M_i *of dimension* $n-1$ *and intersecting each local surface* M_i *of dimension* $n-2$ *is contained in* K *and forms in* K *a neighborhood of the line* h_0. *Conversely, if almost every line* h_0 *of the complex* K *has a neighborhood of the indicated form, then* K *is an admissible complex in general position.*

Theorem 4. *For* $n = 3$, *every admissible complex of lines is either the complex of lines tangent to some two-dimensional algebraic submanifold, or the complex of lines intersecting some algebraic curve.*

3. Plancherel theorem for the group $GL(n, C)$.

Here we shall consider the complex of planes that arises in the derivation of the Plancherel formula on the group $GL(n, C)$. We shall treat matrices $x \in GL(n, C)$ as points of the space C^{n^2}. Consider in $GL(n, C)$ the family K of lower triangular unipotent matrices. It is obvious that these surfaces are surfaces given by parametric equations $x = x_1^{-1} z x_2$, where x_1, x_2 are fixed matrices, and the "parameter" z runs through lower triangular $n(n-1)/2$-dimensional planes in C^{n^2}. It is easy to verify that the manifold K of these planes is a complex. We shall prove here that the complex K is admissible, and at the same time compute the corresponding differential form $\varkappa_\beta\varphi$. In view of the homogeneity of the space $GL(n,C)$, it suffices to consider the case $\beta = e$, where e is the identity matrix.

Introduce the notation: x_+ is the matrix obtained from x by replacing by zeros all elements x_{pq} lying below the main diagonal; $x_- = x - x_+$. Any $n(n-1)/2$-dimensional plane in C^{n^2} (with the exception of a submanifold of planes of lower dimension) can be given by the following equation:

$$x_+ = \sum_{p>q} a^{pq} x_{pq} + \beta, \tag{3}$$

where a^{pq}, β are upper triangular matrices. Take the elements of the matrices a^{pq}, β as local coordinates on the manifold of all $n(n-1)/2$-dimensional planes in C^{n^2}. In these coordinates the form $\varkappa_e\varphi$, defined in (2), has the form

$$\varkappa_e\varphi = \bigwedge_{p>q}\left(\sum_{i\le j} D_{ij}\, da_{ij}^{pq}\right) \wedge \bigwedge_{p>q}\left(\sum_{i\le j} \overline{D}_{ij}\, d\overline{a}_{ij}^{pq}\right)\varphi\Big|_{\beta=e}$$

$$= \bigwedge_{p>q} \operatorname{tr}(D'\,d\alpha^{pq}) \wedge \bigwedge_{p>q} \operatorname{tr}(\bar{D}'\,d\bar{\alpha}^{pq})\,\varphi\Big|_{\beta=e},$$

where $D = \|D_{ij}\|$, $\bar{D} = \|\bar{D}_{ij}\|$; $D_{ij} = \partial/\partial\beta_{ij}$, $\bar{D}_{ij} = \partial/\partial\bar{\beta}_{ij}$ for $i \le j$; $D_{ij} = \bar{D}_{ij} = 0$ for $i > j$; D' denotes the transposed matrix. This form is defined on the manifold of planes passing through e.

Let us find the restriction $i^*\chi_e\varphi$ of the form $\chi_e\varphi$ to the submanifold of planes passing through e and belonging to the complex K. These planes (with the exception of a submanifold of planes of lower dimension) are given by the parametric equations $x = \zeta^{-1}z\xi$, in which the "parameter" is the matrix z, and ξ is an upper triangular unipotent matrix defining the plane. We reduce the equations of the planes to the form (3). Eliminating the "parameter" z from the equations, we obtain

$$x_+ = \zeta^{-1}(\zeta x_- \xi^{-1})_+ \xi + e.$$

Thus

$$\alpha^{pq} = \zeta^{-1}(\zeta e^{pq}\zeta^{-1})_+\zeta,$$

where e^{pq} is the matrix whose entry at the intersection of the p-th row and the q-th column is 1, and whose remaining entries are zeros. Introduce the notation $\zeta'^{-1}D\zeta' = F$, $d\zeta\cdot\zeta^{-1} = u$. Then after elementary transformations we obtain:

$$\bigwedge_{p>q}\operatorname{tr}(D'\,d\alpha^{pq}) = \bigwedge_{q<p}\left(\zeta^{-1}(F_+u' - u'F_+)_-\zeta\right)_{qp} = \bigwedge_{q<p}(uF_+' - F_+'u)_{qp} * .$$

Since

$$(uF'_+ - F'_+u)_{qp} = (F_{pp} - F_{qq})u_{qp} + \sum_{i>p} F_{pi}u_{qi} - \sum_{i<q} F_{iq}u_{ip},$$

it is easy to verify from this that

$$\bigwedge_{q<p}(uF'_+ - F'_+u)_{qp} = \prod_{q<p}(F_{pp} - F_{qq}) \cdot \bigwedge_{q<p} u_{qp} = \prod_{q<p}(F_{pp} - F_{qq}) \cdot \bigwedge_{q<p} d\zeta_{qp}.$$

As a result we obtain

$$i^*\chi_e\varphi = \prod_{q<p}(F_{pp} - F_{qq})(\bar{F}_{pp} - \bar{F}_{qq})\varphi\big|_{\beta=e} \cdot \bigwedge_{q<p} d\zeta_{qp} \wedge \bigwedge_{q<p} d\bar{\zeta}_{qp}. \qquad (4)$$

It remains to verify that the operators F_{pp} belong to the tangent space to the complex K, and consequently the right-hand side of equality (4) depends only on $i^*\varphi$. For this, consider the manifold of planes of the complex K defined by the parametric equations $x = \zeta^{-1}z\delta\zeta$, where δ is a diagonal matrix with diagonal entries $\delta_1, \dots, \delta_n$. In the form (3), the equations of these planes have the form

$$x_+ = \zeta^{-1}(\zeta x_- \zeta^{-1})_+\zeta + \zeta^{-1}\delta\zeta,$$

whence $\beta = \zeta^{-1}\delta\zeta$. Therefore,

$$\frac{\partial}{\partial\delta_p} = \sum_{i,j}(\zeta^{-1})_{ip}\zeta_{pj}\frac{\partial}{\partial\beta_{ij}},$$

i.e.

$$\partial/\partial\delta_p = F_{pp}.$$

Thus, finally, we obtain

$$i^*\chi_e\varphi = \prod_{q<p}\left(\frac{\partial}{\partial\delta_p} - \frac{\partial}{\partial\delta_q}\right)\left(\frac{\partial}{\partial\bar{\delta}_p} - \frac{\partial}{\partial\bar{\delta}_q}\right)(i^*\varphi)\big|_{\delta=e} \cdot \bigwedge_{q<p} d\zeta_{qp} \wedge \bigwedge_{q<p} d\bar{\zeta}_{qp}.$$

This form coincides with the differential form obtained in ([3], [4]). To find the coefficient c_γ in the inversion formula, one must, according to ([2]), take a basis $\gamma_1, \dots, \gamma_s$ in the homology group $H_{2k}(G_e)$, $k = n(n-1)/2$, consisting of Schubert submanifolds, where γ_1 is the Euler cycle (i.e. the manifold of planes lying in a fixed $(k+1)$-dimensional plane). Let $\gamma = G_e \cap K = \sum a_i\gamma_i$; then $c_\gamma = (2i)^k\pi^{2k}a_1$. By simple arguments (for example, computing the intersection index) one can show that $a_1 = n!$.

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* By x'_- (x'_+) is denoted the matrix transposed to x_- (respectively, x_+).

Note: Figure translations are in progress. See original paper for figures.