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Abstract

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ON EQUATIONS AND BOUNDARY-VALUE PROBLEMS IN THE THEORY OF TWO-PHASE FILTRATION FLOWS IN A POROUS MEDIUM

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The equations of isothermal filtration of a two-phase flow of incompressible immiscible fluids in an inhomogeneous isotropic nondeformable porous medium, taking into account phase permeabilities, capillary forces, and differences in densities, follow from the relations ([1])

$$\mathbf{v}_i = -K_i \nabla \Phi_i; \tag{1}$$

$$s_1 + s_2 = 1 \quad (s_2 \equiv s); \tag{2}$$

$$\Phi_1 - \Phi_2 = p_c - \Delta\rho \cdot gh; \tag{3}$$

$$\nabla \mathbf{v}_1 + m\,\partial(1-s)/\partial t = 0, \qquad \nabla \mathbf{v}_2 + m\,\partial s/\partial t = 0, \tag{4}$$

where

$$K_i = kf_i(s)/\mu_i, \qquad \Phi_i = p_i + \rho_i gh, \qquad \Delta\rho = \rho_2 - \rho_1, \qquad i = 1, 2. \tag{5}$$

Here the indices 1 and 2 refer, respectively, to the displaced and displacing fluids; $\mathbf{v}_i$ is the filtration velocity; Φ_i is the head; s_i is the saturation of the pore volume by the i-th fluid, in fractions of unity; p_i is the hydrodynamic pressure in the i-th phase; $m(x, y, z)$ is the local open porosity; μ_i is the absolute viscosity; $p_c(k, s)$ is the interphase capillary pressure*; $k(x, y, z)$ is the local absolute permeability; $f_i(s)$ is the phase (relative) permeability; ρ_i is the fluid density; g is the acceleration of gravity; $h(x, y, z)$ is the elevation of the point (x, y, z) above the plane of zero gravitational potential.

From (1)–(5) we obtain

$$\nabla \cdot (K \nabla \Phi_i) = \nabla \cdot \{(K - K_i) \nabla (p_c - \Delta\rho \cdot gh)\}; \tag{6}$$

$$m\, \partial s/\partial t = (-1)^i \nabla \cdot (K_i \nabla \Phi_i), \tag{7}$$

where $K = K_1 + K_2,\ i = 1, 2$.

Let us choose as the unknown functions the pressure in either of the phases p_i and the saturation s. Then from (6)–(7), taking (5) into account, we have

$$\nabla \cdot (K \nabla p_i) = U_i; \tag{8}$$

$$m\, \partial s/\partial t = -\nabla \cdot \{(K - K_i)[(p_c)'_k \nabla k + (p_c)'_s \nabla s]\} + W_i, \tag{9}$$

where

$$U_i = (-1)^{i-1} \nabla \cdot \{(K - K_i)[(p_c)'_k \nabla k + (p_c)'_s \nabla s]\} - \nabla \cdot (L \nabla h),$$

$$L = g(\rho_1 K_1 + \rho_2 K_2), \qquad (p_c)'_k = \partial p_c/\partial k, \qquad (p_c)'_s = \partial p_c/\partial s, \tag{10}$$

$$W_i = (-1)^{i-1} \nabla \cdot [(K - K_i) \nabla p_i + (L - g\rho_{iK} i) \nabla h].$$

* In view of the small variability of the porosity m over the reservoir, we do not take into account the dependence of p_c on m; we also henceforth assume the wetting angle ϑ and the interfacial tension σ to be constant.

Putting $i = 1$ in (8)–(10), we arrive at a system of two equations with respect to the pressure in the displaced phase p_1 and the saturation s, and for $i = 2$, at a system with respect to the pressure in the displacing phase p_2 and s. Since $K > 0$, equation (8) is of elliptic type with respect to p_i, while, in view of $(p_c)'_s < 0$, equation (9) is of parabolic type with respect to s.

If capillary and gravitational effects are neglected and, in addition, the porous medium is assumed homogeneous ($p_c = 0,\ \nabla h = 0,\ k = \text{const}$), then from (8)–(9) we obtain a system equivalent to that considered in (2).

We proceed to the formulation of boundary-value problems. To simplify the exposition we restrict ourselves to the plane case. Let there be a filtration domain G with boundary Γ, where $\Gamma \subset G$. Introduce, for a point $(x, y) \in \Gamma$, a local orthogonal coordinate system (τ, ν), where τ is the direction tangent to Γ, and ν is the normal to Γ. Since it follows from (3) that

$$p_1 - p_2 = p_c(k, s), \tag{11}$$

then, because $\Gamma \subset G$, on Γ

$$\partial p_1/\partial\nu - \partial p_2/\partial\nu = (p_c)'_k\,\partial k/\partial\nu + (p_c)'_s\,\partial s/\partial\nu. \tag{12}$$

It should also be borne in mind that the boundary conditions on Γ in filtration problems are naturally formulated in $\Phi_i,\ p_i$ or $\partial\Phi_i/\partial\nu,\ \partial p_i/\partial\nu$.

1. Let $p_1(\tau, t)$ be prescribed on Γ, where τ is the arc abscissa of the point $(x, y) \in \Gamma^*$. Then, having the initial saturation distribution $s(x, y, 0)$, we pose the Cauchy problem for the system of equations (8)–(9) with $i = 1$: determine $p_1(x, y, t)$ and $s(x, y, t)$.

In view of the initial condition $s(x, y, 0) = s^0$, we see that the right-hand side of equation (8), as well as the function K, are known at $t = 0$. Consequently, we have the first boundary-value problem for the Poisson-type equation (8): $\nabla\cdot(K^0\nabla p_1) = U_1^0$. It has a unique solution $p_1(x, y, 0) = p_1^0$. Having obtained it, we find $p_2(x, y, 0) = p_1(x, y, 0) - p_c(k, s^0)$. With known p_1^0 and p_2^0, from relation (12) we arrive at a boundary condition of the second kind for s on Γ:

$$\frac{\partial s}{\partial\nu} = \frac{1}{(p_c)'_s}\left[\frac{\partial p_1}{\partial\nu} - \frac{\partial p_2}{\partial\nu} - (p_c)'_k\frac{\partial k}{\partial\nu}\right]. \tag{13}$$

Now, taking into account the available initial distribution s^0 and the nonlinear boundary condition (13), we are able to determine from (9) $s(x, y, \Delta t) = s^1$, since all coefficients and free terms in the right-hand side of (9) and (13) are already known. The next time step begins with the solution of equation (8) under the boundary condition $p_1(\tau, \Delta t)$ and the found saturation distribution s^1, and so on.

2. On Γ, $p_2(\tau, t)$ is prescribed. Then we use the system (8)–(9) with $i = 2$, for which we carry out the same reasoning as in problem 1 for $i = 1$, or, passing to p_1 by means of (11), we again arrive at problem 1.

3. On Γ, $\partial p_1/\partial\nu$ is prescribed. Then, applying the system (8)–(9) with $i = 1$, we arrive, for p_1, at the second boundary-value problem for an equation of Poisson type. To obtain a unique solution it is sufficient to prescribe p_1 as a function of time at one point of the domain G. The determination of s^1 is carried out with the aid of the boundary condition (13). In all other respects the reasoning is analogous to that carried out in problem 1.

4. The situation is completely analogous for the system (8)–(9) with $i = 2$ in the case where $\partial p_2/\partial\nu$ is prescribed on Γ.

5. On part of Γ, p_1 is prescribed, and on the remainder $\partial p_1/\partial \nu$. Then we have a mixed boundary-value problem for equation (8) ($i = 1$) with respect to the unknown p_1^0; after solving it, we find s^1 as before.

* Hereafter, in all problems, the constants μ_i, ρ_i and the functions $m(x, y)$, $k(x, y)$, $h(x, y)$, $f_i(s)$, $p_c(k, s)$ are assumed known, as are the initial distribution $s(x, y, 0)$ and the total flux $q(\tau, t) = K_1 \partial \Phi_1/\partial \gamma - K_2 \partial \Phi_2/\partial \gamma$.

Thus, for the numerical solution of the Cauchy problem for the system (8)–(9) one may propose the following algorithm: a) from the distribution s^n known at the n-th step and the corresponding boundary conditions for p_i (or $\partial p_i/\partial \nu$), determine p_i^{n+1} at the $(n + 1)$-st step from equation (8); b) from the found p_i^{n+1}, the boundary condition (13), and the known distribution s^n, determine s^{n+1} from equation (9) and pass to the next time step.

A check balance relation for the system (8)–(9) is

$$\iint\limits_G m \frac{\partial s}{\partial t} \, dx \, dy = (-1)^i \int\limits_\Gamma K_i \left(\frac{\partial p_i}{\partial \nu} + g\rho_i \frac{\partial h}{\partial \nu} \right) d\tau. \tag{14}$$

It is clear that everything set forth for plane problems is carried over without difficulty to three-dimensional flows as well.

It is necessary to note the following two circumstances. First, the approach introduced completely removes the widely known difficulties associated with the correct formulation of boundary-value problems of two-phase filtration when attempts are made to prescribe on the boundary Γ a first-kind boundary condition for the saturation s. It has been shown above that, under boundary conditions formulated for Φ_i, p_i, or their normal derivatives, the problem of determining s can always be posed with boundary conditions of the second kind. Second, when the saturation s reaches the lower ($\underline{s}$) or upper ($\overline{s}$) limiting value, determined by the behavior of the phase-permeability curves, respectively the second or the first phase becomes immobile ($K_2(\underline{s}) = 0,\ K_1(\overline{s}) = 0$). In the corresponding zones of the filtration region there occurs a degeneration of the parabolic equation (9), which, however, does not prevent the solution of the boundary-value problem for this equation in the whole region. As is easy to see, in the zones $K_1 = 0,\ K_2 = 0$ (changing with time) the saturation s preserves the constant values $\overline{s}$ and s^*.

Let us indicate one more formulation of the problem, applicable when there exists a stream function ψ, i.e., for plane and axisymmetric spatial flows. For the plane problem we introduce ψ by the relations

$$\partial\psi/\partial y = K_1 \, \partial\Phi_1/\partial x + K_2 \, \partial\Phi_2/\partial x, \qquad -\partial\psi/\partial x = K_1 \, \partial\Phi_1/\partial y + K_2 \, \partial\Phi_2/\partial y. \tag{15}$$

After transformations we arrive at the following system of equations with respect to ψ, s, of the same type as the systems (8)–(9):

$$\nabla\cdot(K^{-1}\nabla\psi) = LK^{-1}D\,[h, \ln(LK^{-1})] + MK^{-1}\{(p_c)'_{kD}\,[k, \ln(K_1K_2^{-1})]+ \\ +(p_c)'_{sD}\,[s, \ln(K_1K_2^{-1})]\}, \qquad (16)$$

$$m\partial s/\partial t = \nabla\cdot\{M[\Delta\rho\cdot g\nabla h - (p_c)'_k\nabla k - (p_c)'_s\nabla s]\} + D\,[N,\psi], \qquad (17)$$

where

$$D[u,v] = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}, \qquad M = \frac{K_1K_2}{K}, \qquad N = \frac{K_2-K_1}{2K}.$$

The system (16)–(17) is applicable not under arbitrary boundary conditions for Φ_i, p_i, but only when they can be reformulated as boundary conditions for ψ. The algorithm for solving boundary-value problems for the system in ψ, s is essentially analogous to that described earlier for the systems (8)–(9).

We note that the boundary-value problems formulated above are especially convenient to solve on hybrid (analog-digital) computing machines. In this case the boundary-value problem for p_i or ψ is solved on the grid of an electronic integrator, while the determination of s and the recalculation of all coefficients and functions at each time step are assigned to the digital computer. At the same time, solution of the system proposed in (2) is difficult because of the existence of a frontal

* Here it is assumed that $\underline{s} \le s(x,y,0) \le \overline{s}$.

a jump in saturation; in systems (8)–(9) and (16)–(17) the real capillary forces, which hinder the formation of jumps, are taken into account, as a result of which methods of through computation over the entire domain G can be used.

In conclusion we touch upon the system of equations of two-phase filtration proposed in (1). In our notation it has the form

$$\nabla\cdot(K_i\nabla\Phi_i) = (-1)^i\frac{m}{(p_c)'_s}\left(\frac{\partial\Phi_1}{\partial t} - \frac{\partial\Phi_2}{\partial t}\right). \qquad (18)$$

This system is not a system of Cauchy–Kovalevskaya type. Douglas et al. (1), in the numerical solution of system (18), were forced to use a finite-difference approximation essentially equivalent not to system (18), but to another one, supplemented in the right-hand sides respectively by the terms $\varepsilon_i^n\partial\Phi_i/\partial t$. At each time level iterations were then carried out with $\varepsilon_i^0 > \varepsilon_i^1 > \varepsilon_i^2 > \cdots$. It can be shown that the "closer" the system being solved was to (18), the worse was the conditioning of the system of difference equations. In (1) another form of the equations was also used, obtained by passing to the variables $P = \frac{1}{2}(\Phi_1 + \Phi_2)$ and $R = \frac{1}{2}(\Phi_1 - \Phi_2)$, which, like system (16)–(17), is not applicable under arbitrary boundary conditions.

The formulation of problems in p_i, s (systems (8)–(9)) is free from the indicated shortcomings and restrictions. The approach set forth extends to a number of phases in the flow greater than two.

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Note: Figure translations are in progress. See original paper for figures.