ON THE DIMENSION OF THE PRODUCT OF ONE-DIMENSIONAL BICOMPACTA

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Abstract

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ON THE DIMENSION OF THE PRODUCT OF ONE-DIMENSIONAL BICOMPACTA

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In this paper we shall prove the following two theorems:

Theorem 1. Let

$$P = \prod_{i=1}^{n} X_i,$$

where X_i is a bicompactum and $\operatorname{Ind} X_i = 1$ for every i. Then

$$\dim P = \operatorname{ind} P = \operatorname{Ind} P = n.$$

Theorem 2. Let

$$P = \prod_{i=1}^{\infty} X_i,$$

where X_i is an ordered continuum for every i. Then P is strongly infinite-dimensional.

Before proving the theorems, we recall some known definitions and formulate several lemmas.

Definition 1. Let X be a normal space. A cover $\alpha = \{A_1, \dots, A_n\}$ of the space X is called a **partition** if $A_i = [\operatorname{Int} A_i]$ and $\operatorname{Int} A_i \cap \operatorname{Int} A_j = \emptyset$ for $i \neq j$.

We introduce a new definition.

Definition 2. Let $\alpha = \{A_1, \dots, A_n\}$ be a partition of the space X; then by $\operatorname{bd} \alpha$ we shall denote the set

$$X \setminus \bigcup_{i=1}^{n} \operatorname{Int} A_i.$$

Lemma 1. Let X be a normal space and $\operatorname{Ind} X \leq 1$. Let $F \subset X$ be a closed set, $\operatorname{Ind} F \leq 0$, and let $\gamma = \{\Gamma_1, \dots, \Gamma_n\}$ be a cover of F by sets open in it. Then into any open cover $\omega = \{O_1, \dots, O_m\}$ of the space X one can inscribe a partition $\alpha = \{A_1, \dots, A_l\}$ such that: 1) $\operatorname{Ind} \operatorname{bd} \alpha \leq 0$ and $\operatorname{bd} \alpha \cap F = \emptyset$; 2) $\alpha \wedge F$

is a partition of F, consisting of sets open-and-closed in F and inscribed in the cover γ.

Lemma 2. Let X be a normal space. If into every finite open cover of the space X one can inscribe a partition α such that $\operatorname{Ind} \operatorname{bd} \alpha \le n-1$, then $\operatorname{Ind} X \le n$.

Proof of Theorem 1.

A. We first prove the inequality $\operatorname{Ind} P \le n$.

The proof will be by induction. We shall show that 1) into any cover ω of the bicompactum P one can inscribe a partition α^1, such that 2) into any cover of $\operatorname{bd} \alpha^1$ one can inscribe a partition α^2 of the set $\operatorname{bd} \alpha^1$, with $\operatorname{bd} \alpha^2$ such that, 3) into any ⋯and so on, that $n-1$) into any cover of $\operatorname{bd} \alpha^{n-1}$ one can inscribe a partition α^n such that $\operatorname{Ind} \operatorname{bd} \alpha^n \le 0$. Then we shall obtain that $\operatorname{Ind} \operatorname{bd} \alpha^{n-1} \le 1$, $\operatorname{Ind} \operatorname{bd} \alpha^1 \le n-1$, and $\operatorname{Ind} P \le n$.

1) Let $\omega^1 = \{O_1^1, \dots, O_s^1\}$ be an arbitrary open cover of the space P. Then into ω^1 one can inscribe a cover

$$\tilde{\omega}^1 = \{\omega_1^1 \times \dots \times \omega_n^1\},$$

where ω_i^1 is a cover of the space X_i, and $\tilde{\omega}^1$ is a family of open sets, each of which is the product of n sets, one from each ω_i^1 $(i = 1, 2, \dots, n)$. Into each cover ω_i^1 we inscribe a partition α_i^1 such that $\operatorname{Ind} \operatorname{bd} \alpha_i^1 \le 0$; then the partition

$$\alpha^1 = \{\alpha_1^1 \times \dots \times \alpha_n^1\}$$

will be inscribed in the cover $\tilde{\omega}^1$, and

$$\operatorname{bd} \alpha^1 = \bigcup_{i=1}^{n} \prod_{\substack{i=1 \\ i \ne i_1}}^{n} X_i \times \operatorname{bd} \alpha_{i_1}^1.$$

2) Let now $\omega^2 = \{O_1^2, \dots, O_{i_{\omega^3}}^2\}$ be an arbitrary covering of $\operatorname{Fr} \alpha^1$. Denote

$$X^{i_1} = \prod_{\substack{i=1 \\ i \ne i_1}}^{n} X_i \times \operatorname{Fr} \alpha_{i_1}^1.$$

On each X^l $(l = 1, 2, \dots, n)$ inscribe, in the covering $X^l \wedge \omega^2$, the covering $\gamma_l = \{\gamma_1^l \times \dots \times \gamma_n^l\}$. Then on X_i we shall have $(n-1)$ coverings and one covering on $\operatorname{Fr} \alpha_i^1$.

Take on X_i such a covering $\tilde{\gamma}_i$ that it is inscribed in all the coverings on X_i, and $\tilde{\gamma}_i \wedge \operatorname{Fr} \alpha_i^1$ is inscribed in the covering γ_i^i. Now, by Lemma 2, inscribe in the covering $\tilde{\gamma}_i$ such a partition $\tilde{\alpha}_i^2$ that $\tilde{\alpha}_i^2$ is inscribed in the covering $\tilde{\gamma}_i$ and a) $\operatorname{Ind} \operatorname{Fr} \tilde{\alpha}_i^2 \leqslant 0$, $\operatorname{Fr} \tilde{\alpha}_i^2 \cap \operatorname{Fr} \alpha_i^1 = \emptyset$; b) $\tilde{\alpha}_i^2 \wedge \operatorname{Fr} \alpha_i^1$ is a partition into sets open-and-closed in $\operatorname{Fr} \alpha_i^1$, which we denote by β_i. Take now on X_i the partition $\alpha_i^2 = \tilde{\alpha}_i^2 \wedge \alpha_i^1$. If on each X^l we take the partition

$$\tilde{\alpha}^l = \prod_{\substack{i=1\\ i\neq l}}^{n} \alpha_i^2 \times \beta_l,$$

then

$$\alpha^2 = \bigvee_{l=1}^{n} \tilde{\alpha}^l$$

(where the sign $\bigvee_{l=1}^{n}$ denotes the sum in the sense of taking the whole collection of elements of the partitions $\tilde{\alpha}^l$, and not in the sense of the union of sets from $\tilde{\alpha}^l$) will be a partition inscribed in the covering ω^2.

Let us prove this last assertion. Indeed:

a) the inscription of the system of sets α^2 in the covering ω^2 is fulfilled by construction.

b) Let A be an arbitrary element of some $\tilde{\alpha}^l$; then $\mathrm{Int}_{X^l} A$ contains points from $\mathrm{Fr}\,\alpha_l^1$, but contains no points from $\mathrm{Fr}\,\alpha_i^1$ $(i \neq l,\ i = 1, 2, \dots, n)$. Therefore

$$\mathrm{Int}_{X^l} A \cap X^i = \emptyset \quad (i \neq l,\ i = 1, 2, \dots, n),$$

since X^i contains the factor $\mathrm{Fr}\,\alpha_i^1$ $(i \neq l,\ i = 1, 2, \dots, n)$. Consequently, the $\mathrm{Int}_{X^l} A$ are open in $\mathrm{Fr}\,\alpha^1$, and these sets are pairwise disjoint; the sum of their closures will give the entire set $\mathrm{Fr}\,\alpha^1$.

The assertion is proved. Thus, α^2 is a partition of $\mathrm{Fr}\,\alpha^1$, inscribed in the covering ω^2, and

$$\mathrm{Fr}\,\alpha^2 = \bigcup_{i_1=1}^{n} \bigcup_{\substack{i_2=1\\ i_2\neq i_1}}^{n} \prod_{\substack{i=1\\ i\neq i_1, i_2}}^{n} X_i \times \mathrm{Fr}\,\alpha_{i_1}^1 \times \mathrm{Fr}\,\alpha_{i_2}^2.$$

Suppose that at the $(k-1)$-st step we already have

$$\mathrm{Fr}\,\alpha^{k-1} = \bigcup_{i_1=1}^{n} \bigcup_{\substack{i_2=1\\ i_2\neq i_1}}^{n} \cdots \bigcup_{\substack{i_{k-1}=1\\ i_{k-1}\neq i_1,\dots,i_{k-2}}}^{n} \prod_{\substack{i=1\\ i\neq i_1,\dots,i_{k-1}}}^{n} X_i \times \mathrm{Fr}\,\alpha_{i_1}^1 \times \mathrm{Fr}\,\alpha_{i_2}^2 \times \cdots$$

$$\cdots \times \mathrm{Fr}\,\alpha_{i_{k-1}}^{k-1},$$

where

$$\operatorname{Fr}\alpha_i^1 \subset \operatorname{Fr}\alpha_i^2 \subset \cdots \subset \operatorname{Fr}\alpha_i^{k-1}$$

and $\operatorname{Fr}\alpha_i^l$ is open-and-closed in $\operatorname{Fr}\alpha_i^{l+1}$, $1 \leqslant l \leqslant k-2$.

Denote

$$X^{i_1 i_2 \cdots i_{k-1}} = \bigcup_{P(1,2,\dots,k-1)} \prod_{\substack{i=1\\ i\neq i_1,\dots,i_{k-1}}}^{n} X_i \times \operatorname{Fr}\alpha_{i_1}^{j_1} \times \cdots \times \operatorname{Fr}\alpha_{i_{k-1}}^{j_{k-1}}$$

($i \neq i_1, \dots, i_{k-1}$; $(j_1, \dots, j_{k-1})$ is an element of the permutation of $1, 2, \dots, k-1$),

$$X^{i_1 \cdots i_{k-1}}_{j_1 \cdots j_{k-1}} = \prod_{\substack{i=1\\ i\neq i_1,\dots,i_{k-1}}}^{n} X_i \times \operatorname{Fr}\alpha_{i_1}^{j_1} \times \cdots \times \operatorname{Fr}\alpha_{i_{k-1}}^{j_{k-1}},$$

then

$$X^{i_1 i_2 \cdots i_{k-1}} = \bigcup_{P(1,2,\dots,k-1)} X^{i_1 \cdots i_{k-1}}_{j_1 \cdots j_{k-1}}, \qquad \operatorname{Fr}\alpha^{k-1} = \bigcup_{C_n^{i_1 \cdots i_{k-1}}} X^{i_1 i_2 \cdots i_{k-1}}.$$

We carry out the k-th step. Let ω^k be an arbitrary cover by open sets of the set $\operatorname{gr}\alpha^{k-1}$. On each $X^{i_1 \cdots i_{k-1}}_{j_1 \cdots j_{k-1}}$ we inscribe in the cover $\omega^k \wedge X^{i_1 \cdots i_{k-1}}_{j_1 \cdots j_{k-1}}$ a cover $\gamma^{i_1 \cdots i_{k-1}}_{j_1 \cdots j_{k-1}} = \{\gamma_1 \times \cdots \times \gamma_n\}$. Then on X_i we obtain a certain number of covers both on $\operatorname{gr}\alpha_i^1, \operatorname{gr}\alpha_i^2, \dots, \operatorname{gr}\alpha_i^{k-1}$. Take on X_i a cover $\tilde{\gamma}_i$ by open sets such that it is inscribed in all the given covers on X_i, and $\tilde{\gamma}_i \wedge \operatorname{gr}\alpha_i^j$ $(j = 1, 2, \dots, k-1)$ is inscribed in all the given covers on $\operatorname{gr}\alpha_i^j$.

Now, by Lemma 2, inscribe in the cover $\tilde{\gamma}_i$ such a partition $\tilde{\alpha}_i^k$ that $\tilde{\alpha}_i^k$ is inscribed in $\tilde{\gamma}_i$ and a) $\operatorname{Ind}\operatorname{gr}\tilde{\alpha}_i^k \leq 0$, $\operatorname{gr}\tilde{\alpha}_i^k \cap \operatorname{gr}\alpha_i^{k-1} \neq \emptyset$; b) $\tilde{\alpha}_i^k \wedge \operatorname{gr}\alpha_i^{k-1}$ is a partition of $\operatorname{gr}\alpha_i^{k-1}$ into sets open-and-closed in $\operatorname{gr}\alpha_i^{k-1}$, which we denote by β_i^{k-1}; moreover, we require that, if an element of the partition β_i^{k-1} meets $\operatorname{gr}\alpha_i^j$ $(1 \leq j \leq k-1)$, then it is contained in it, and we denote $\beta_i^j = \beta_i^{k-1} \wedge \operatorname{gr}\alpha_i^j$ $(1 \leq j \leq k-1)$. Next take the partition $\alpha_i^k = \tilde{\alpha}_i^k \wedge \alpha_i^{k-1}$; then on each $X^{i_1 \cdots i_{k-1}}_{j_1 \cdots j_{k-1}}$ we obtain the partition

$$\alpha^{i_1 \cdots i_{k-1}}_{j_1 \cdots j_{k-1}} = \{\alpha_1^k \times \cdots \times \alpha_{i_1-1}^k \times \beta_{i_1}^{j_1} \times \alpha_{i_1+1}^k \times \cdots \times \alpha_{i_{k-1}-1}^k \times \beta_{i_{k-1}}^{j_{k-1}} \times \alpha_{i_{k-1}+1}^k \times \cdots \\ \cdots \times \alpha_n^k\}.$$

If an element of the partition $\alpha^{i_1 \dots k-1}_{j_1 \dots j_{k-1}}$ meets some bicompactum among the constituent bicompacta $X^{i_1 i_2 \dots i_{k-1}}$, then it belongs entirely to this bicompactum and is an element of a partition on it.

Thus, if one takes

$$\alpha^{i_1 \dots i_{k-1}} = \bigvee_{P(1,\dots,k-1)} \alpha^{i_1 \dots i_{k-1}}_{j_1 \dots j_{k-1}},$$

where the sum is understood in the sense that, if two elements from different summands meet, then they are identified as one. Then $\alpha^{i_1 \dots i_{k-1}}$ is a partition of the bicompactum $X^{i_1 \dots i_{k-1}}$ by virtue of the properties described above of the partitions on $X^{i_1 \dots i_{k-1}}_{j_1 \dots j_{k-1}}$ $(j_1 \dots j_{k-1} \in P(1, 2, \dots, k-1))$. Further, if one takes

$$\alpha^k = \bigvee_{C_n^{i_1 \dots i_{k-1}}} \alpha^{i_1 \dots i_{k-1}},$$

then we obtain that α^k is a partition on $\operatorname{gr} \alpha^{k-1}$. Indeed, if A is an element of the partition $\alpha^{i_1 \dots i_{k-1}}$, then $\operatorname{Int} A \cap X^{j_1 \dots j_{k-1}} = \emptyset$ (for $j_1 \dots j_{k-1} \neq i_1 \dots i_{k-1}$, i.e., if they differ in at least one index). We have

$$\operatorname{gr} \alpha^k = \bigcup_{i=1}^{n} \bigcup_{\substack{i_2=1 \\ i_2 \neq i_1}}^{n} \cdots \bigcup_{\substack{i_k=1 \\ i_k \neq i_1,\dots,i_{k-1}}}^{n} \prod_{i=1}^{n} X_i \times \operatorname{gr} \alpha^k_{i_1} \times \cdots \times \operatorname{gr} \alpha^k_{i_k}$$

(where $i \neq i_1, \dots, i_k$) or

$$\operatorname{gr} \alpha^k = \bigcup_{C_n^{i_1 \cdots i_k}} \bigcup_{P(1,2,\dots,k)} \prod_{i=1}^{n} X_i \times \operatorname{gr} \alpha^{j_1}_{i_1} \times \cdots \times \operatorname{gr} \alpha^{j_k}_{i_k} \quad (i \neq i_1, \dots, i_k).$$

For $k = n$ we obtain that $\operatorname{gr} \alpha^n$ is the sum of a finite number of zero-dimensional bicompacta. Consequently, $\operatorname{Ind} \operatorname{gr} \alpha^n \leq 0$, while $\operatorname{Ind} \operatorname{gr} \alpha^{n-1} \leq 1$, hence $\operatorname{Ind} \operatorname{gr} \alpha^1 \leq n-1$. Thus the inequality $\operatorname{Ind} P \leq n$ is proved.

B. We shall now prove the inequality $\dim P \geq n$.

This inequality for our case follows from the work of Cohen ([2]), where he introduces the notion of cohomological dimension, which goes back to the work of P. S. Aleksandrov ([1]).

Cohen denotes his dimension by cd and proves in ([2]) the following theorems, which we shall formulate for the case when the space X is a bicompactum (we retain Cohen' s numbering).

Theorem 7.2. For any bicompactum X, $\operatorname{cd}(X) \leq \dim X$.

Theorem 5.8. If X is a bicompactum and $\operatorname{ind} X = 1$, then $\operatorname{cd}(X) = 1$.

Theorem 6.5. If $\operatorname{cd}(X) = n$ and $\operatorname{cd}(Y) = 1$, X, Y are bicompacta, then any of the following conditions is sufficient for $\operatorname{cd}(X \times Y) = n + 1$:
a) $n = 0$, b) $\operatorname{ind} X = n$, c) $\operatorname{ind} Y = 1$.

It follows from these theorems that $\dim P \geq n$.

The theorem of P. S. Aleksandrov stating that, for bicompacta, $\dim X \leq \operatorname{ind} X$, completes the proof of our Theorem 1.

Before proving our Theorem 2, let us recall the definition of a strongly infinite-dimensional space, given by P. S. Aleksandrov.

Definition. A space X is called **strongly infinite-dimensional** if there exists a countable number of pairs of closed sets $\{C_i, C_i'\}$ such that $C_i \cap C_i' = \emptyset$, and such that the intersection of any closed sets B_i, each of which is a partition* for the pair (C_i, C_i'), is nonempty $(i = 1, 2, \ldots, n, \ldots)$.

Proof of Theorem 2. Every ordered bicompactum has a minimal and a maximal point.

Denote the end points of the bicompactum X_i by i_0 and i_1. Then, as the pairs (C_i, C_i'), one may take the sets

$$C_i = \prod_{\substack{j=1 \\ j \neq i}}^{\infty} X_j \times i_0, \qquad C_i' = \prod_{\substack{j=1 \\ j \neq i}}^{\infty} X_j \times i_1.$$

Now let B_i be an arbitrary closed partition for the pair (C_i, C_i') $(i = 1, 2, \ldots, n)$. We shall prove that $\bigcap_{i=1}^{n} B_i \neq \emptyset$. Since P is a bicompactum, it suffices to prove that the intersection of any finite number of the sets B_i is nonempty. This follows from the proof of assertion 1 in my paper ([5]). Theorem 2 is proved.

Theorem 2 is a positive answer to a question posed to me by Yu. M. Smirnov during my talk at the International Congress of Mathematicians in Moscow in 1966.

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* A partition in the space X between C and C' ($C \cap C' = \emptyset$) is a set B such that $X \setminus B = G \cup H$, where G and H are disjoint open subsets of X such that $C \subset G$, and $C' \subset H$.

Note: Figure translations are in progress. See original paper for figures.