θ-PROXIMITIES AND θ-ABSOLUTES

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Introduction. This article is closely connected with the paper ([7]), almost all of whose results are contained here in a more general form. The concept of θ-proximity, introduced in ([7]), is extended here to the case of all Hausdorff spaces. It turns out that, with the aid of θ-proximity, one can describe all irreducible perfect and even θ-perfect (continuity is replaced by θ-continuity in the sense of S. V. Fomin ([9])) mappings of completely regular spaces. If proximities on a given space correspond, as Yu. M. Smirnov showed ([6]), to bicompact extensions of this space (bicompact extensions of homeomorphic preimages of this space), then θ-proximities on a given space correspond to bicompact extensions of completely regular preimages of this space under θ-perfect irreducible mappings.

The principal method of the work is the method of centered systems of open sets. This method was first considered by P. S. Aleksandrov ([1]), who, with its aid, gave a very transparent construction of the maximal bicompact extension of a completely regular space. Yu. M. Smirnov ([6]), generalizing P. S. Aleksandrov' s method, used the concept of proximity to construct all bicompact extensions of completely regular spaces. In an entirely different direction, P. S. Aleksandrov' s method was generalized in the works of S. V. Fomin and S. Iliadis (see ([2,3,8])), in which, in particular, the maximal bicompact extension of a maximal θ-perfect irreducible preimage of a given Hausdorff space was constructed. In the present work the method of centered systems of open sets, with the aid of the concept of θ-proximity, makes it possible to obtain a result of which all those listed above are special cases.

The concept of a θ-proximally continuous mapping is introduced. In this connection the θ-mappings introduced in ([7]) are perfect irreducible θ-proximally continuous mappings.

A result analogous to Yu. M. Smirnov's theorem on the extension of δ-continuous mappings to the corresponding bicompact extensions of proximity spaces is proved. It is shown that in every class of θ-spaces mapped onto one another by means of multivalued regular θ-mappings there exists a projective space (a θ-absolute). At the same time, V. I. Ponomarev' s theorem on the absolute of a topological space can be formulated in terms of θ-proximity. In this formulation it turns out to be a special case of the theorem on the θ-absolute. The concept of

θ-extremal disconnectedness, generalizing the concept of extremal disconnectedness, is also introduced. There is a natural connection between θ-absolutes and θ-extremally disconnected θ-spaces. The θ-absolutes are precisely the ordinary proximity spaces, and only they. From this point of view, ordinary proximities are distinguished among all θ-proximities in the same way as extremally disconnected spaces are distinguished among all Hausdorff spaces.

Definitions and statements of theorems. We shall say that a θ-**proximity** is given on a topological space X if, for any two subsets A and B of X, one of the following two conditions holds:

relations: $A\theta B$ (A is θ-near to B) or $A\bar{\theta}B$ (A is θ-far from B), with the following axioms satisfied:

I. $A\theta B \Rightarrow B\theta A$.

II. $A\theta B_i,\ i = 1,2 \Longleftrightarrow A\bar{\theta}\,(B_1 \cup B_2)$.

III. $\emptyset\bar{\theta}X$.

IV. $\{x\}\theta\{y\} \Rightarrow x = y$.

V. $A\bar{\theta}B \Rightarrow$ there exists a canonically open set C such that
$\bar{C}\bar{\theta}B$ and $A\bar{\theta}(X \setminus [C])$.

VI. $\{x\}\theta A \Rightarrow$ any neighborhoods of the point x and of the set A intersect.*

It follows from the axioms that θ-nearnesses exist only on Hausdorff spaces. It turns out (see the corollary to Theorem 1) that on every Hausdorff space there exist θ-nearnesses. The most general example of a θ-nearness is given by

Theorem 1. Let $f : Z \to X$ be a θ-perfect irreducible mapping of a completely regular space Z onto the space X, and let bZ be a bicompact extension of the space Z. Then bZ induces on X the following θ-nearness:

$$A\bar{\theta}B \Longleftrightarrow [f^{-1}A]_{bZ} \cap [f^{-1}B]_{bZ} = \emptyset.$$

Corollary. On every Hausdorff space there exists a θ-nearness.

This follows from the fact that every Hausdorff space has an absolute which is a completely regular space.

Theorem 2. Every θ-nearness on a Hausdorff space X determines a completely regular space X_θ, a θ-perfect irreducible mapping $\pi_{X_\theta} : X_\theta \to X$ onto X, and a bicompact extension $b_\theta X_\theta$, which induces (in the sense of Theorem 1) the given θ-nearness.

Theorem 3. Let $f_1 : Z_1 \to X$ and $f_2 : Z_2 \to X$ be θ-perfect irreducible mappings of completely regular spaces Z_1 and Z_2 onto X. If the bicompact extensions b_1Z_1 and b_2Z_2 induce on X the same θ-nearness, then there exists a homeomorphism $g : b_1Z_1 \to b_2Z_2$ such that $gZ_1 = Z_2$ and $f_1 = f_2g$.

Corollary 1. The set of θ-nearnesses on a bicompactum is in one-to-one correspondence with the set of perfect irreducible mappings onto this bicompactum.**

Corollary 2. Every θ-nearness on an extremally disconnected space is an ordinary nearness.

Corollary 3. If a θ-nearness on the space X is an ordinary nearness, then the mapping $\pi_{X_\theta} : X_\theta \to X$ is a homeomorphism, and the bicompactum $b_\theta X_\theta$ is a bicompact extension of the proximity space X.

Let X and Y be two θ-spaces, i.e., two topological spaces on each of which a θ-nearness is given. A mapping $f : X \to Y$ will be called **θ-nearness-continuous** if two conditions are satisfied:

1°. $A, B \subset Y,\ A\bar{\theta}B \Rightarrow f^{-1}A\bar{\theta}f^{-1}B$.

2°. $A, B \subset Y,\ A\theta B \Rightarrow \langle f^{-1}[A]\rangle\theta\langle f^{-1}[B]\rangle$.***

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* The first four axioms are the axioms of a generalized proximity space. Axiom V is a weakening of the normality axiom for a proximity space, compatible with a topological space. Axiom VI shows the connection of θ-nearness with topology. For regular spaces axiom VI is equivalent to axiom VI′: $\{x\}\theta A \Rightarrow x \in [A]$ (axiom IV of paper (7)).

** Two mappings $f_1 : Z_1 \to X$ and $f_2 : Z_2 \to X$ are identified if there exists a homeomorphism $g : Z_1 \to Z_2$ such that $f_1 = f_2 g$.

*** $\langle C\rangle$ denotes the interior of the set C.

Lemma. Every θ-proximity continuous mapping is θ-continuous.

We shall call a θ-space X a **regular θ-space** if the space X is a regular space.

Theorem 4. Let $f : X \to Y$ be a θ-proximity continuous mapping of a θ-space X onto a regular θ-space Y. Then there exists a continuous mapping $f_\theta : b_\theta X_\theta \to b_\theta Y_\theta$ of the bicompactum $b_\theta X_\theta$ onto the bicompactum $b_\theta Y_\theta$ such that $f_\theta X_\theta \subset Y_\theta$ and the diagram is commutative

$$\begin{array}{ccc} X_\theta & \xrightarrow{f'_\theta} & Y_\theta \\ \pi_{X_\theta}\downarrow & & \downarrow\pi_{Y_\theta} \\ X & \xrightarrow{f} & Y \end{array}$$

where f'_θ is the restriction of the mapping f_θ to X_θ.

Remark. The restriction of the mapping f_θ to X_θ is not, generally speaking, a mapping onto Y_θ, but is only a mapping onto some dense subset of the space Y_θ.

Theorem 5. If a θ-proximity continuous mapping $X \to Y$ of a θ-space X onto a θ-space Y is perfect, then $f_\theta X_\theta = Y_\theta$.

It has been possible to get rid of the condition of regularity of the θ-space Y only by imposing restrictions on the mapping f.

Theorem 6. Let $f : X \to Y$ be a θ-proximity continuous mapping of a θ-space X onto a θ-space Y. Suppose that the mapping f is closed and irreducible. Then there exists a continuous irreducible mapping f_θ of the bicompactum $b_\theta X_\theta$ onto the bicompactum $b_\theta Y_\theta$ such that $f_\theta X_\theta \subset Y_\theta$ and the diagram is commutative

$$\begin{array}{ccc} X_\theta & \xrightarrow{f'_\theta} & Y_\theta \\ \pi_{X_\theta} \downarrow & & \downarrow \pi_{Y_\theta} \\ X & \xrightarrow{f} & Y \end{array}$$

If, moreover, the mapping f is bicompact, then $f^{-1}Y_\theta = X_\theta$, and the mapping f'_θ is perfect and irreducible.

Corollary. Every multivalued θ-mapping* $f : X \leftarrow Z \to Y$ of a θ-space X onto a θ-space Y generates such a multivalued irreducible mapping $f_\theta : b_\theta X_\theta \leftarrow b_\theta Z_\theta \to b_\theta Y_\theta$ that $\varphi^{-1} X_\theta = Z_\theta = \psi^{-1} Y_\theta$ and $f \pi_{X_\theta} = \pi_{Y_\theta} f'_\theta$.

A θ-proximity continuous mapping $f : X \to Y$ will be called a **θ-mapping** if f is θ-perfect and irreducible.** In accordance with (4), a θ-mapping $f : X \to Y$ will be called a **regular θ-mapping** if $A, B \subset Y$, $A\theta B \Rightarrow f^{-1}A\theta f^{-1}B$.

Definition. A θ-space X^θ is called a **θ-absolute** of the θ-space X if there exists a canonical projection $\pi_X : X^\theta \to X$, which is a regular θ-mapping, such that for every regular θ-mapping $f : Z \to X$ onto X there exists a regular θ-mapping $g : X^\theta \to Z$ onto Z such that $\pi_{X^\theta} = fg$.

* A multivalued mapping $f : X \to Y$ of θ-spaces will be called a **multivalued θ-mapping** if there exist a θ-space Z and such closed, irreducible, bicompact and θ-proximity continuous mappings $f_X : Z \to X$, $f_Y : Z \to Y$, that $f = f_{Y f} X^{-1}$. For regular spaces this definition coincides with the definition given in (7).

** For regular spaces this definition coincides with the definition given in (7).

A space X endowed with the maximal θ-nearness θ_a (two sets are θ_a-far if and only if they have disjoint neighborhoods) will be called a **maximal θ-space**. It can be shown that every θ-perfect irreducible mapping $f : X \to Y$ of a maximal θ-space X onto a maximal θ-space Y is a regular θ-mapping. After this, the theorem of V. I. Ponomarev ([5]) on the absolute of a topological space can be reformulated as follows:

Every maximal θ-space X has an absolute X^θ with respect to the class of regular θ-mappings of maximal θ-spaces.

The following theorem is a generalization of V. I. Ponomarev’ s theorem.

Theorem 7. *Every θ-space X has a θ-absolute X^θ, which is unique in the sense that any two θ-absolutes X_1^θ and X_2^θ of the space X are connected by a θ-equimorphism $h : X_1^\theta \to X_2^\theta$, satisfying the condition $\pi_{X_2^\theta} h = \pi_{X_1^\theta}$. For every multivalued regular θ-mapping $f : X \to Y$ onto Y there exists a θ-equimorphism $f^\theta : X^\theta \to Y^\theta$ such that*
$f = \pi_{Y^\theta} f^\theta \pi_{X^\theta}^{-1}$. Every regular θ-mapping onto a θ-absolute is a θ-equimorphism.

Corollary 1. *In order that a θ-space X be a θ-absolute, it is necessary and sufficient that the θ-nearness on X be a nearness.*

As is known, a space is extremally disconnected if and only if, for any two disjoint open subsets of it, their closures do not intersect. In terms of θ-nearness this means that any two θ-far sets in the maximal θ-nearness have θ-far closures. What has been said above justifies the following

Definition. A θ-space X is called **θ-extremally disconnected** if, in it, any two θ-far sets have θ-far closures.

Corollary 2. *A θ-space X is a θ-absolute if and only if it is θ-extremally disconnected.*

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* A mapping $f : X \to Y$ of a θ-space X onto a θ-space Y is called a **θ-equimorphism** if it is one-to-one and θ-nearness-continuous in both directions.