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Fig. 1

Figure 1: Fig. 1

Abstract

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MATHEMATICS

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ON EXPANSION IN EIGENFUNCTIONS OF THE SCHRÖDINGER OPERATOR WITH A STRONGLY SINGULAR POTENTIAL

(Presented by Academician A. N. Tikhonov, 3 I 1968)

Consider the behavior of a quantum particle in the potential field shown in Fig. 1. It is clear that if the height of the barrier V_0 is sufficiently large, then the wave function will differ little from the wave function of a particle in the potential field shown by the dotted line in Fig. 1. Our article is devoted to a rigorous proof of this fact. The usual methods of perturbation theory are not suitable here, since the perturbation of the potential in the present case is infinite. A. A. Samarskii⁽¹⁾, using the example of the Helmholtz equation, showed that in such problems it is necessary to introduce a certain nontrivial metric for the perturbation. We shall proceed in the same way.

Fig. 1

Let R_N be an N -dimensional Euclidean space ($N \geq 3$), Δ the Laplace operator, $V(x)$ a scalar function (potential), $\Omega = \{x; V(x) = +\infty\}$. Below we assume that the function $V(x)$ satisfies two conditions:

- 1) each function $V_M(x) = \min\{V(x), M\}$, for $M \geq 0$, is nonnegative and locally satisfies the Hölder condition;
- 2) there exist constants $\alpha > 0$, $C < \infty$, and $R < \infty$, independent of M , such that for all x lying outside the ball of radius R , the estimate

$$V(x) < C|x|^{-N-\alpha}$$

holds.

Let $G_M(t)$ be the Green function of the Cauchy problem

$$u(x, 0) = u_0(x); \quad \partial u / \partial t = \Delta u - V_M(x)u, \quad t > 0, \quad x \in R_N; \quad u(x, t) \in L^\infty.$$

Lemma. *The semigroup of operators $G_M(t)$, as $M \rightarrow \infty$, converges in the uniform operator topology of the space $[L^p \rightarrow L^p, 1 < p < \infty]$ to the operator $G(t)$; the operators $G(t)$ form a semigroup in t in $L^p, 1 \leq p \leq \infty$.*

It can be verified that in $L^p(R_N \setminus \Omega)$ the semigroup $G(t)$ is a semigroup of class C_0 . Let A be the infinitesimal operator of the semigroup $G(t)$, A_M the infinitesimal operator of the semigroup $G_M(t)$, the operator P be defined by the formula $(Pu)(x) = u(x), x \in R_N \setminus \Omega, (Pu)(x) = 0, x \in \Omega$, and the operators H and H_M be defined by the formulas $Hu = -APu, H_M u = -A_M u$. It is proved that H and H_M are self-adjoint operators; by $E(\lambda, H)$ and $E(\lambda, H_M)$ we denote the spectral functions of the operators H and H_M .

The spectral function $E(\lambda, H)$ is constructed with the aid of the solution of the scattering problem $u(x, k)$, i.e. the solution of the equation $Hu = \lambda u, u \in L^\infty, x \in R_N$, which can be represented as the sum of two functions

$$u(x, k) = \exp(ikx) + \varphi(x, k), \quad k^2 = \lambda,$$

where the function $\varphi(x, k)$ satisfies the radiation conditions:

$$\begin{aligned} \varphi(x, k) &= O(|x|^{(1-N)/2}), & (\partial/\partial|x| - i|k|)\varphi(x, k) &= o(|x|^{(1-N)/2}), \\ & & |x| &\rightarrow \infty. \end{aligned}$$

(The functions $u_M(x, k)$ and $\varphi_M(x, k)$ are defined analogously with the aid of the operator H_M .)

Theorem 1. 1) Let $f(x)$ be an arbitrary function from L^2 , and let $f_n(x) \in L^1 \cap L^2$ be a sequence converging in $L^2(R_N \setminus \Omega)$ to the function $f(x)$. To each function $f_n(x)$ we associate the function

$$(\tilde{f}_n)(k) = \int u(x, k) f_n(x) dx.$$

It is asserted that the sequence $(\tilde{f}_n)(k)$ converges in L^2 to a certain function $(\tilde{f})(k)$, and the function $(\tilde{f})(k)$ does not depend on the choice of the sequence $f_n(x)$, but is determined only by the function $f(x)$.

2) Let $\{\lambda_i\}$ be the set of eigenvalues of the discrete spectrum of the operator H , and let $\psi(x, \lambda_i)$ be the corresponding eigenfunctions. If $\lambda \notin \{\lambda_i\}$, then the operator $E(\lambda, H)$ can be computed by the formulas

$$\begin{aligned} \langle f_1, E(\lambda, H) f_2 \rangle &= (2\pi)^{-N} \int_{|k|^2 \leq \lambda} (\tilde{f}_1)^*(k) (\tilde{f}_2)(k) dk + \sum_{\lambda_i < \lambda} (f_1)_i^* (f_2)_i, \\ (E(\lambda, H) f)(x) &= (2\pi)^{-N} \int_{|k|^2 \leq \lambda} u^*(x, k) (\tilde{f})(k) dk + \sum_{\lambda_i < \lambda} f_i \psi(x, \lambda_i), \quad (1) \end{aligned}$$

where $f_i = \langle \psi(x, \lambda_i), f \rangle$ is the scalar product of the functions $\psi(x, \lambda_i)$ and $f(x)$ in L^2 . The integral over k in formula (1) is understood in the

following sense: let $\tilde{f}_n(k)$ be a sequence of functions converging in L^2 to $(\tilde{f})(k)$, with each of the functions $\tilde{f}_n(k)$ equal to zero in a certain neighborhood, depending on n , of the set $\{k, k^2 \in \{\lambda_i\}\}$. For each n the integral

$$I_n(x) = \int_{|k|^2 \leq \lambda} u^*(x, k) \tilde{f}_n(k) dR$$

exists as a Lebesgue integral. It is asserted that the sequence $I_n(x)$ converges in L^2 , and its limit depends only on $(\tilde{f})(k)$.

3) The transformation

$$f \rightarrow Uf = [(2\pi)^{-N/2}(\tilde{f})(k); \{f_i\}], \quad f_i = \langle \psi(x, \lambda_i), f \rangle$$

is a one-to-one unitary transformation of $L^2(R_N \setminus \Omega)$ onto the orthogonal sum of the spaces $L^2(R_N) \oplus l_m^2$, where m is the cardinality of the set $\{\lambda_i\}$. The inverse transformation is given by the formula

$$f(x) = U^{-1}[(2\pi)^{-N/2}(\tilde{f})(k); \{f_i\}] = \lim_{\lambda \rightarrow \infty} \left((2\pi)^{-N} \int_{|k|^2 \leq \lambda} u^*(x, k) (\tilde{f})(k) dk + \sum_{\lambda_i < \lambda} f_i \psi(x, \lambda_i) \right). \quad (2)$$

In formula (2) the integral over k is understood in the sense indicated above, and the limit with respect to λ is in the L^2 metric.

The spectral expansion of the operator H_M is constructed in a completely analogous way. The existence and uniqueness of the functions $u(x, k)$ were proved in work (2); there we also showed that the functions $\varphi(x, k)$ and $\varphi_M(x, k)$, which are solutions of the equations

$$\varphi(x, k) = T^+(\lambda)(\exp(iky) + \varphi(y, k)),$$

$$\varphi_M(x, k) = T_M^+(\lambda)(\exp(iky) + \varphi_M(y, k)), \quad \lambda = k^2,$$

where the operators $T^+(\lambda)$ and $T_M^+(\lambda)$ are integral operators completely continuous in the metric $[L^p \rightarrow L^p, 2N/(N-1) < p < \infty]$, and that the function $\psi(x, \lambda_i) \in L^2$ is an eigenfunction of the operator H if and only if it satisfies the equation

$$T^+(\lambda_i)\psi = \psi.$$

The fact that, in the uniform operator topology of the space $[L^p \rightarrow L^p, 2N/(N-1) < p < \infty]$, the equality

$$T^+(\lambda_j) = \lim_{\rho_j(\lambda, M) \rightarrow 0} T_M^+(\lambda), \quad \text{where } \rho_j(\lambda, M) = \sqrt{|\lambda - \lambda_j|^2 + M^{-2}},$$

holds allows us to prove the theorem:

Theorem 2. There exists a $\delta_j > 0$ such that for all λ, M such that

$$0 < \rho_j(\lambda, M) < \delta_j,$$

the function $u_M(x, k)$ can be represented in the form

$$u_M(x, k) = \exp(ikx) + \mu_j(\lambda, M)(1 - \mu_j(\lambda, M))^{-1}\psi(x, \lambda_j, \lambda, M) \times \\ \times (\psi^*)_0(k, \lambda_j, \lambda, M) + x_j(x, k, M), \quad \lambda = k^2,$$

where the function $\psi(x, \lambda_j, \lambda, M)$ is an eigenfunction of the equation

$$T_M^+(\lambda)\psi = \mu_j(\lambda, M)\psi,$$

$(\psi^*)_0(k, \lambda_j, \lambda, M)$ is the Fourier transform of the eigenfunction of the equation

$$T_M^+(\lambda)^*\psi^* = \mu_j\psi^*,$$

and

$$\lim_{\rho_j(\lambda, M) \rightarrow 0} \mu_j(\lambda, M) = 1, \quad \lim_{\rho_j(\lambda, M) \rightarrow 0} \|\psi(x, \lambda_j, \lambda, M) - \psi(x, \lambda_j)\|_{L^p} = 0, \\ \frac{2N}{N-1} < p < \infty,$$

and the function $x_j(x, k, M)$ is bounded in the metric L^p , $2N/(N-1) < p < \infty$, as $\rho_j(\lambda, M) \rightarrow 0$.

We now formulate the main result of our work, which is a consequence of Theorems 1 and 2. Suppose that $F(\lambda)$ is a continuous function bounded on $[0, \infty]$, $D(M) = F(H) - F(H_M)$. The function $D(M)f$ can be represented as the sum of two terms

$$(D(M)f)(x) = \int_0^\lambda F(\lambda') d\{E(\lambda', H) - E(\lambda', H_M)\}f + \int_\lambda^\infty F(\lambda') d\{E(\lambda', H) - \\ - E(\lambda', H_M)\}f = D_1(M, \lambda)f + D_2(M, \lambda)f, \quad \lambda \notin \{\lambda_i\}.$$

Theorem 3. If $f(x) \in \dot{W}_2^1(R_N \setminus \Omega) \cap L^1 \cap L^\infty$, and all eigenvalues $\{\lambda_i\}$ are simple, then there exists a constant $C_0(f)$, independent of M, F , and λ , such that

$$\|D_2(M, \lambda)f\|_2^2 < C_0(f)(1 + \lambda)^{-1} \max_{\xi > \lambda} |F(\xi)|^2,$$

and the function $D_1(M, \lambda)f$ satisfies the equality

$$\text{s.l.} \quad \lim_{M \rightarrow \infty} D_1(M, \lambda)f = 0.$$

Moreover, whatever bounded closed set S may be,

$$\lim_{M \rightarrow \infty} \|D_1(M, \lambda)f\|_{L^1(S)} = 0.$$

A consequence of Theorem 3 is the equality

$$\text{s.l. } \lim_{M \rightarrow \infty} F(H_M)f = F(H)f.$$

Let Ω_1 be the largest open connected set contained in $R_N \setminus \Omega$ and containing the points $\{x, |x| > R\}$, and let

$$\Omega_2 = R_N \setminus (\Omega_1 \cup \Omega).$$

Theorem 4. Suppose that the function $V(x)$ satisfies the following conditions: $\text{mes } \Omega_2 > 0$, $\rho(\Omega_1, \Omega_2) > 0$, the function $F(\lambda)$ is measurable and bounded on $[0, \infty)$, and the function $f(x)$ is equal to zero outside the set Ω_2 and

$$\int f(x)(Hf)(x) dx = E_0 < \infty.$$

Let ε be an arbitrary positive number, let the number $\lambda(\varepsilon) \notin \{\lambda_j\}$ satisfy the inequality

$$\lambda(\varepsilon) > 4\varepsilon^{-2}E_0 \max |F(\xi)|^2.$$

Then there exists a number $M(\varepsilon) < \infty$ and a function $\sigma(M) \rightarrow +0$ as $M \rightarrow \infty$, such that for all $M \geq M(\varepsilon)$ the inequality

$$\left\| F(H_M)f - \sum_{\lambda_j < \lambda(\varepsilon)} (2\pi)^{-N} \int_{\lambda_j - \sigma(M) \leq k^2 \leq \lambda_j + \sigma(M)} u_M^*(x, k)(\tilde{f})_M(k)F(k^2) dk \right\|_2 < \varepsilon$$

holds.

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CITED LITERATURE

1. A. A. Samarskii, DAN, 63, No. 6, 631 (1948).
2. A. A. Arsen' ev, DAN, 178, No. 6 (1968).

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