SPHERICAL SOUND WAVES IN RIEMANNIAN SPACE

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Abstract

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K. P. STANYUKOVICH

SPHERICAL SOUND WAVES IN RIEMANNIAN SPACE

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1. Basic equations for spherical waves. The equations describing spherically symmetric motions of a medium in its own gravitational field have the form (1)

$$\frac{1}{c\theta^2}[Au_t+uu_r]-\frac{\omega^2}{c^2}\left[(\ln V)_r+\frac{Au}{c^2}(\ln V)_t\right]+\frac{1}{2}\left[\nu_r+\frac{Au}{c^2}\lambda_t\right]=\frac{\theta^2T^0\sigma_r}{W};\quad(1,1)$$

$$-\left[A(\ln V)_t+u(\ln V)_r\right]+\frac{1}{\theta^2}\left[u_r+\frac{Au}{c^2}u_t\right]+\frac{1}{2}[A\lambda_t+u\nu_r]=0;\qquad(1,2)$$

$$A\sigma_t+u\sigma_r=0;\qquad(1,3)$$

$$\nu_r=\frac{u^2}{c^2}\left[\lambda_r+\frac{e^\lambda-1}{2}+\varkappa pre^\lambda\right]+\frac{e^\lambda-1}{2}+\varkappa pre^\lambda\qquad(1,4)$$

or

$$A\lambda_t+u\lambda_r+u\left[\frac{e^\lambda-1}{2}+\varkappa pre^\lambda\right]=0;\qquad(1,4a)$$

$$A(1+u^2/c^2)\lambda_t+u(\lambda+\nu)_r=0\qquad(1,5)$$

or

$$(re^{-\lambda})_r=1-\frac{\varkappa r^2}{\theta^2}\left(\varepsilon+p\frac{u^2}{c^2}\right).\qquad(1,5a)$$

This system of 5 independent equations uniquely determines u, V, σ, λ and ν for a given equation of state $p=p(V,\sigma)$ and with use of the identity

$\partial(p, V)/\partial(T^0, \sigma) = 1$; here p is the pressure; T^0 is the temperature; V is the specific volume; σ is the entropy; $u = (u_\alpha u^\alpha)^{1/2}$ is the 3-velocity; $\theta^2 = 1 - u^2/c^2$; $\omega^2/c^2 = -(\partial \ln W/\partial \ln V)_\sigma$ is the speed of sound; $W = pV + E = pV + \rho V c^2$ is the heat content; $\rho c^2 = \varepsilon$ is the energy density, with $A = e^{(\lambda-\nu):2}$, $dE = d(\rho V c^2) = T^0 d\sigma - p\,dV$, where λ and σ determine the metric of the centrally symmetric field:

$$-ds^2 = -c^2 dt^2 e^\nu + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{1,6}$$

In the case of equilibrium, when $u = 0$, we shall have:

$$V p_r/W + \nu_r/2 = 0; \tag{1,7}$$

$$\lambda_t = 0, \qquad \sigma_t = 0, \qquad p_t = 0;$$

$$\nu_r = (e^\lambda - 1)/r + \varkappa p r e^\lambda; \tag{1,8}$$

$$d(re^{-\lambda})/dr = 1 - \varkappa r^2 \varepsilon. \tag{1,9}$$

Eliminating λ and ν from (1,7), (1,8), and (1,9), we arrive at the well-known Oppenheimer–Volkoff equation (2)

$$\frac{d}{dr}\left[\frac{r(1 + \varkappa r^2 p)}{1 - r\,dp/(p+\varepsilon)\,dr}\right] = 1 - \varkappa r^2 \varepsilon. \tag{1,10}$$

We shall take the parameters determined by these equations as the zeroth approximation. For $u \neq 0$, when $u/c \ll 1$, we shall have:

$$Au_t - \omega^2(\ln V)_r + \frac{c^2}{2}\left[\nu_r + \frac{Au}{c^2}\lambda_t\right] = \frac{T^0\sigma_r}{W}; \tag{1,11}$$

$$-A(\ln V)_t + u_r + 2u/r + \frac{1}{2}[A\lambda_t + u\nu_r] = 0; \tag{1,12}$$

$$A\sigma_t + u\sigma_r = 0; \tag{1,13}$$

$$\nu_r = (e^\lambda - 1)/r + \varkappa p r e^\lambda; \tag{1,14}$$

$$A\lambda_t + u(\lambda + \nu)_r = 0; \tag{1,15}$$

or

$$\partial(re^{-\lambda})/\partial r = 1 - \varkappa r^2 \varepsilon. \tag{1,15a}$$

2. Sound waves in Einstein space. Consider the simplest case $p_0 = \text{const}$, then $\nu_r = 0$, whence, without loss of generality, one may put $\nu_0 = 0$; in this case $e^{-\lambda_0} = 1 + \varkappa r^2 p_0$, or

$$\lambda_0 = -\ln(1 + \varkappa r^2 p_0), \quad \sigma_0 = \text{const}, \quad \partial\lambda_0/\partial r = -2\varkappa r p_0/(1 + \varkappa r^2 p_0). \tag{2,1}$$

Further we shall have

$$(1 + \varkappa r^2 p_0)\left(1 + \frac{2\varkappa p_0 r^2}{1 + \varkappa r^2 p_0}\right) = 1 + 3\varkappa r^2 p_0 = 1 - \varkappa r^2 \varepsilon_0, \tag{2,2}$$

which gives the equation of state

$$3p_0 + \varepsilon_0 = 0, \tag{2,3}$$

which is the equation of state of the spherical Einstein world. Here $p_0 = p_\mathrm{m} + p_\mathrm{f}$; $\varepsilon_0 = \varepsilon_\mathrm{m} + \varepsilon_\mathrm{f}$; $\varepsilon_\mathrm{f} = 3p_\mathrm{f}$, where the indices f and m refer to the values of ε and p for the field and matter, respectively. In this case $\varepsilon_\mathrm{f} = -(3p_\mathrm{m} + \varepsilon_\mathrm{m})/2$, $\varepsilon_0 = (\varepsilon_\mathrm{m} - 3p_\mathrm{m})/2$. Obviously, ε_f should be understood as the density of the negative energy of the gravitational field itself. In this case there is no need to introduce the so-called λ-term into the Einstein equations. Instead, the energy and pressure of the field are introduced. Further, it is evident that $\varkappa\varepsilon_0 = 3/a^2$; $\varkappa p_0 = -1/a^2$;

$$1 + \varkappa p_0 r^2 = 1 - r^2/a^2 = e^{-\lambda}; \tag{2,4}$$

$$-ds^2 = -c^2 dt^2 + \frac{dr}{1 - r^2/a^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{2,5}$$

i.e. the metric of the Einstein world.

For $u/c < 1$ we shall have $\lambda = \lambda_0 + \Delta\lambda = -\ln(1 + \varkappa r^2 p_0) + \Delta\lambda$, $\nu = \Delta\nu$, $V = V_0 + \Delta V$, $p = p_0 + \Delta p$, $\varepsilon = \varepsilon_0 + \Delta\varepsilon$, $\sigma = \sigma_0 + \Delta\sigma$. (If $\lambda = -\nu$, $e^\nu = 1 + \varkappa r^2 p_0/3$, then we obtain that $\varepsilon_0 + p_0 = 0$; this equation of state corresponds to the de Sitter metric.)

$$-ds^2 = -c^2 dt^2\left(1 - \frac{r^2}{a^2}\right) + \frac{dr^2}{1 - r^2/a^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{2,6}$$

The investigation of small perturbations for the de Sitter world is analogous to what we do for the Einstein world.

The basic equations (1,11), (1,13) take the form

$$Au_t - \omega_0^2 \frac{\Delta V_r}{V_0} + \frac{c^2}{2}\Delta\nu_r = \frac{T^0}{W}\sigma_r, \quad A\frac{\Delta V_t}{V_0} + u_r + \frac{2u}{r} + \frac{A\Delta\lambda_t}{2} = 0, \qquad (2,7)$$

$$\Delta\sigma_t = 0,$$

i.e.

$$\Delta\sigma = \Delta\sigma(r). \qquad (2,8)$$

In this case

$$A = e^{(\lambda_0 + \Delta\lambda - \Delta\nu)/2} = \frac{1}{\sqrt{1 - r^2/a_0^2}}\left[1 + \frac{\Delta\lambda - \Delta\nu}{2}\right]. \qquad (2,9)$$

For the time being let us consider the more particular case when $\Delta\sigma = 0$, i.e. purely isentropic sound waves.

Equations (2,6) and (2,7) then take the form

$$\frac{u_t}{\sqrt{1 - r^2/a^2}} - \frac{\omega_0^2}{V_0}(\Delta V)_r + \frac{c^2}{2}\Delta\nu_r = 0; \qquad (2,10)$$

$$-\frac{(\Delta\nu)_t}{\sqrt{1 - r^2/a^2}} + u_r + \frac{2u}{r} + \frac{\Delta\lambda_t}{2\sqrt{1 - r^2/a^2}} = 0. \qquad (2,11)$$

Further, equation (1,14) gives

$$\nu_r = \frac{\Delta\lambda}{r} + \frac{\varkappa r\Delta p}{1 - r^2/a^2} = \frac{\Delta\lambda}{r} - \frac{r\Delta p}{a^2 p_0(1 - r^2/a^2)}, \qquad (2,12)$$

and equation (1.15), which we shall write in the form

$$\frac{\Delta\lambda_t}{\sqrt{1 - r^2/a^2}} = \frac{2\chi r p_0 u}{1 - r^2/a^2},$$

immediately determines

$$\Delta\lambda_t = -\frac{2ru}{a^2\sqrt{1-r^2/a^2}}. \tag{2.13}$$

Now equations (2.10) and (2.11) can be written in the form

$$\frac{u_t}{\sqrt{1-r^2/a^2}} - \frac{\omega_0^2}{v_0}(\Delta v)_r + \frac{c^2}{2}\left[\frac{\Delta\lambda}{r} - \frac{r\Delta p}{a^2 p_0}\left(1-\frac{r^2}{a^2}\right)\right] = 0, \tag{2.14}$$

$$-\frac{(\Delta v)_t}{v_0\sqrt{1-r^2/a^2}} + u_r + \frac{2u}{r} - \frac{u_r}{a^2(1-r^2/a^2)} = 0. \tag{2.15}$$

Since

$$\frac{\omega_0}{V_0}\Delta v = V_0\Delta p,$$

we finally write equations (2.14) and (2.15) in the form

$$\frac{u_t}{\sqrt{1-r^2/a^2}} + v_0\Delta p_r + \frac{c^2}{2}\left[\frac{\Delta\lambda}{r} - \frac{r\Delta p}{a^2 p_0(1-r^2/a^2)}\right] = 0; \tag{2.16}$$

$$-\frac{v_0\Delta p_t}{\omega_0^2\sqrt{1-r^2/a^2}} + u_r + \frac{2u}{r} - \frac{ur}{a^2(1-r^2/a^2)} = 0. \tag{2.17}$$

Let $u = \overline{W}f(r)$, where $\overline{W} = \overline{W}(r,t)$; then (2.17) takes the form

$$-\frac{v_0\Delta p_t}{\omega_0^2\sqrt{1-r^2/a^2}} + f\left[\overline{W}_r + \overline{W}\left(\frac{f'}{f} + \frac{2}{r} - \frac{r}{a^2(1-r^2/a^2)}\right)\right] = 0;$$

we set the expression in parentheses equal to zero and determine

$$\frac{df}{f} = -\frac{2dr}{r} + \frac{r\,dr}{a^2(1-r^2/a^2)}, \tag{2.18}$$

whence

$$fr^2\sqrt{1-r^2/a^2} = B = \text{const}, \qquad u = \frac{B\overline{W}}{r^2\sqrt{1-r^2/a^2}}.$$

In this case equation (2.17) can be written in the form

$$r^2 V_0\Delta p_t/B\omega_0^2 + \overline{W}_r = 0, \tag{2.19}$$

whence

$$\Delta p = -\frac{B\omega_0^2}{r^2 v_0}\frac{\partial\psi}{\partial r}; \qquad (2.20)$$

$$\overline{W} = \frac{\partial\psi}{\partial t} = \frac{ur^2}{B}\sqrt{1 - r^2/a^2} \quad \text{or} \quad u = \frac{B}{r^2\sqrt{1 - r^2/a^2}}\frac{\partial\psi}{\partial t}. \qquad (2.21)$$

In this case equation (2.16) takes the form

$$\frac{\partial^2\psi}{\partial t^2} - \left(1 - \frac{r^2}{a^2}\right)\omega_0^2\frac{\partial^2\psi}{\partial t^2} + 2\left(1 - \frac{r^2}{a^2}\right)\frac{\omega_0^2}{r}\frac{\partial\psi}{\partial r} +$$

$$+\frac{c^2}{2}\left[\frac{(1 - r^2/a^2)}{B}(\Delta\lambda)_r + \frac{r\omega_0^2}{a^2 p_0 v_0}\frac{\partial\psi}{\partial r}\right] = 0. \qquad (2.22)$$

From (2.13) we have

$$\sqrt{1 - r^2/a^2}\,\frac{\partial\Delta\lambda}{\partial t} = -\frac{2ur}{a^2} = -\frac{B}{a^2 r\sqrt{1 - r^2/a^2}}\frac{\partial\psi}{\partial t},$$

whence

$$\Delta\lambda = -\frac{B\psi}{(1 - r^2/a^2)ra^2} + \Phi^*(r).$$

Since for $\psi = 0$, $\Delta\lambda = 0$, it follows that $\Phi^*(r) = 0$, and finally

$$\Delta\lambda = -\frac{B\psi}{(1 - r^2/a^2)ra^2}. \qquad (2.23)$$

Now equation (2.22) can be written in the form:

$$\psi_{x^0x^0} + \frac{1}{2}\left[\psi_r\frac{r}{a^2}\frac{\omega_0^2}{p_0 V_0} - \frac{\psi}{a^2}\right] = \left(1 - \frac{r^2}{a^2}\right)\frac{\omega_0^2}{c^2}\left[\psi_{rr} - \frac{2\psi_r}{r}\right], \qquad (2.24)$$

where $x^0 = ct$. We shall assume that the change in pressure and energy density occurs according to the laws of an ultrarelativistic gas; then

$$\Delta p = -\Delta\varepsilon/3, \qquad \omega_0^2/c^2 = \Delta p/\Delta\varepsilon = -1/3, \qquad -p_0 v_0 = c^2/3 = \omega_0^2;$$

in this case we shall have

$$\psi_{x^0x^0} - \frac{1}{2a^2}[-r\psi_r + \psi] + \frac{1}{3}\left[1 - \frac{r^2}{a^2}\right]\left[\psi_{rr} - \frac{2\psi_r}{r}\right] = 0. \tag{2.25}$$

If we introduce a new function F and a new independent variable χ by means of the relation

$$\psi = a\sin\chi\exp\left[ikx^0 + \int\left(F - \frac{5}{4}\operatorname{tg}\chi\right)d\chi\right], \tag{2.26}$$

where $r = a\sin\chi$, then we arrive at the Riccati equation

$$F_\chi + F^2 = -\frac{5}{4\cos^2\chi} + 2\operatorname{ctg}^2\chi + 3k^2a^2 + \frac{1}{16}\operatorname{tg}^2\chi + 3. \tag{2.27}$$

A numerical solution of this equation presents no difficulties.
Analysis of the basic equation (2.24) leads to interesting results. At the center, for $r = 0$,

$$\psi_{x^0x^0} \simeq \frac{\omega_0^2}{c^2}\left(-\frac{2\psi_r}{r}\right) = \frac{2}{3}\frac{\psi_r}{r}. \tag{2.28}$$

At the "periphery," as $r \to a$,

$$\psi_{x^0x^0} \simeq \frac{1}{2a^2}(a\psi_r + \psi). \tag{2.29}$$

In the central regions the amplitude Δp and the velocity value for the diverging wave decrease as r increases and increase for the converging wave; at the "periphery," on the contrary, the amplitude of the converging wave decreases upon convergence, and the amplitude of the diverging wave increases as r tends to a.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.