ON SINGULAR INTEGRAL EQUATIONS IN A CONE

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Abstract

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1. Notation. Let R^n be the n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$; let S^{n-1} be the unit sphere with center at the origin. Denote by $H_s(S^{n-1})$ the space of functions (for $s \geq 0$) defined on S^{n-1} and having generalized derivatives up to order s inclusive, square-summable. As usual, for $s < 0$ set

$$H_s(S^{n-1}) = H^*_{-s}(S^{n-1}).$$

The norm of an element $u \in H_s(S^{n-1})$ will be denoted by $|u|_s$. In the space R^n introduce spherical coordinates (r, ψ). By $H_{s,\alpha}$ we shall denote the space obtained by completing the set of smooth functions v on R^n with compact supports not containing the origin, with respect to the norm

$$\|v\|^2_{s,\alpha} = \int_0^\infty |v|^2_s r^{\alpha+n-1}\, dr.$$

In the one-dimensional case one considers the space $\mathcal{L}_2^\alpha$ with norm

$$\int_{-\infty}^{+\infty} |v(x)|^2 |x|^\alpha\, dx.$$

Denote by $\Phi(\xi)$, $\xi = (\xi_1, \dots, \xi_n)$, a homogeneous function of degree zero, and by F the Fourier transform operator for functions defined on R^n,

$$Fv = \int e^{-i\xi\cdot x} v(x)\, dx, \qquad \xi \cdot x = \xi_1 x_1 + \dots + \xi_{nx} n.$$

Then the singular operator A on R^n with symbol $\Phi(\xi)$ has the form

$$Av = F^{-1}\Phi(\xi) F v. \tag{1}$$

We assume that the symbol $\Phi(\xi)$ is an infinitely differentiable function everywhere except the origin.

2. Boundedness theorems*. Let us first consider the one-dimensional case. Let $\mathfrak{M}_1^\alpha$ be the set of smooth functions with compact supports not containing the origin, satisfying, for $|\alpha| > 1$, the conditions

$$\int_{-\infty}^{+\infty} v(x)x^l\,dx = 0, \tag{2}$$

where $0 \leq l \leq [(\alpha-1)/2]$, if $\alpha > 1$, and $[(\alpha-1)/2] \leq l \leq -1$, if $\alpha < -1$. Here l is an integer, and by $[\beta]$ we denote the integer nearest to β such that $|[\beta]| \leq |\beta|$. Let us note that the set $\mathfrak{M}_1^\alpha$ is dense in $\mathcal{L}_2^\alpha$.

* As Yu. E. Khaikin informed the author, he independently obtained similar theorems by another method and in other terms.

Theorem 1. In order that the singular operator defined on the set $\mathfrak{M}^a$ by formula (1) be bounded in the space $\mathcal{L}_2^a$, it is necessary and sufficient that the number a not be an odd integer.

For $a \in (-1,1)$ the boundedness of the operator (1) in the space $\mathcal{L}_2^a$ was proved earlier by K. I. Babenko ([1]).

Now let us consider singular operators on R^n, $n \geq 2$. First let $a > -n$. Denote by $\mathfrak{M}_+^a$ the set of smooth functions with compact supports satisfying, for $a > n$, the conditions

$$\int v(x)x_1^{l_1}\cdots x_n^{l_n}\,dx = 0, \tag{3}$$

where $l_1, \ldots, l_n$ denote nonnegative integers such that

$$0 \leq l_1 + l_2 + \cdots + l_n \leq [(a-n)/2].$$

The set $\mathfrak{M}_+^a$ is dense in H_s^a.

Theorem 2. In order that the operator defined on the set $\mathfrak{M}_+^a$ by formula (1) be bounded in the space H_s^a, it is necessary and sufficient that the relations

$$(a-n)/2 \neq k, \qquad k = 0, 1, 2, \ldots$$

hold.

For $a \in (-n,n)$ the boundedness of the singular operator in the space H_0^a was proved earlier by E. M. Stein ([2]).

Let now $a < -n$. Denote by $\mathfrak{M}_-^a$ the set of smooth functions with compact supports not containing the origin of coordinates and satisfying the conditions

$$\int_0^\infty v(r,\psi) r^{q-1}\, dr = \sum_{l=0}^{-q} \sum_K c_{lK} Y_{lK}(\psi), \qquad q = 0, -1, \dots, \left[\frac{a+n}{2}\right]. \quad (4)$$

Here $Y_{lK}(\psi)$ denotes the spherical function of order l, and K denotes a collection of integers $(k_0, \dots, k_{n-2})$ such that $l \equiv k_0 \geq k_1 \geq \dots \geq k_{n-2} \geq 0$. Finally, c_{lK} are arbitrary constants. The set $\mathfrak{M}_-^a$ is dense in H_s^a.

Theorem 3. In order that the singular operator defined on the set $\mathfrak{M}^a$ by formula (1) be bounded in the space H_s^a, it is necessary and sufficient that the relations

$$(a+n)/2 \neq k, \qquad k = 0, -1, -2, \dots$$

hold.

3. Formula for the operator. The singular operator A was initially defined on a set dense in H_s^a by formula (1). Denote by $\bar{A}$ the closure of the operator A. The operator $\bar{A}$ is defined on the entire space H_s^a (for a admissible in the sense of the theorems of point 2). However, on functions that do not satisfy conditions (2), (3), or (4), this operator cannot be defined by formula (1). For $n \geq 2$ the formula

$$(\bar{A}u)(r,\varphi) = \int_{-\infty+ih}^{+\infty+ih} r^{i\lambda-1} e^{i\pi n/2} \Gamma(n-1+i\lambda)\Gamma(1-i\lambda)\, d\lambda \times \quad (5)$$

$$\times \int \Phi(\omega)(\varphi\cdot\omega + i0)^{i\lambda-1}\, dS_\omega \int \tilde{v}(\lambda,\psi)(-\omega\psi + i0)^{1-n-i\lambda}\, dS_\psi, \qquad h = \frac{a+n-2}{2}.$$

holds. Here ω, φ, ψ are unit vectors, $(\pm\varphi\cdot\omega + i0)^\mu$ are generalized functions on the sphere, dS is the element of volume of the sphere S^{n-1}, and by $\tilde{v}(\lambda,\psi)$ is denoted the Mellin transform of the function $v(r,\psi)$,

$$\tilde{v}(\lambda,\psi) = \int_0^\infty r^{-i\lambda} v(r,\psi)\, dr.$$

An analogous formula also holds in the one-dimensional case. On the sets $\mathfrak{M}_+^a, \mathfrak{M}_-^a$ the operators defined by formulas (1) and (5) coincide.

4. The first boundary value problem in a cone. Let K be an n-dimensional cone with vertex at the origin, cutting out on the sphere S^{n-1} a domain G with smooth boundary ∂G. Denote by $t(\psi)$ a continuous function given on $\overline{G}$. Let $\{U_j\}$ be a finite sufficiently fine covering of the domain G, and let $\{\tau_j(\psi)\}$ be the associated partition of unity. Following (3), introduce the space $H_{(t)}(G)$ with norm

$$\|u\|^2_{(t)} = \sum_j \|\tau_j u\|^2_{t_j}, \qquad t_j = t(\psi_j), \quad \psi_j \in U_j,$$

where, if $U_j \cap \partial G \neq \emptyset$, then $\psi_j \in U_j \cap \partial G$. By $H^\alpha_{(t)}(K)$ we denote the space of functions defined in K, with norm

$$\|u\|^2_{(t),\alpha} = \int_0^\infty \|u\|^2_{(t)} r^{\alpha+n-1}\, dr.$$

Finally, by $\dot{H}^\alpha_{(t)}(K)$ we denote the closure of the set of smooth functions with compact supports lying inside K. The operator $\overline{A}$ acts as a bounded operator from the space $\dot{H}^\alpha_{(t)}(K)$ into the space $H^\alpha_{(t)}(K)$ for α satisfying the conditions of Theorems 2, 3.

Suppose that the operator $\overline{A}$ is elliptic, i.e., its symbol does not vanish. By the first boundary value problem in the cone K for the operator $\overline{A}$ we shall mean the problem of finding a solution $u \in \dot{H}^\alpha_{(t)}(K)$ of the equation

$$(\overline{A}u)(x) = f(x), \qquad x \in K, \quad f \in H^\alpha_{(t)}(K). \tag{6}$$

Associate in a natural way local coordinates with the point $\psi \in \partial G$ and compute in these coordinates the symbol of the operator $\overline{A}$. Factor the symbol into two factors (see (4)). Denote the factorization index by $\chi(\psi)$ (we assume that in the two-dimensional case the index $\chi(\psi)$ is determined uniquely).

Theorem 4. Let

$$\max_{\psi\in\partial G} |t(\psi) + \chi(\psi)| < \tfrac{1}{2}.$$

Then there exists a unique solution $u \in \dot{H}^\alpha_{(t)}(K)$ of problem (6) for all α, except for some countable set, and for all $f \in H^\alpha_{(t)}(K)$.

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Note: Figure translations are in progress. See original paper for figures.