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Abstract

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On the Problem of Extending Linear Positive Functionals

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In the present paper a number of new theorems on the extension of linear positive functionals are given. The question of extending linear positive functionals in a Banach space with a solid cone was investigated by M. G. Krein ([1]).

1. Consider, in a real Banach space E with cone K ([1]), a linear functional f defined on some subspace $E_f \subset E$. Denote by $\mathcal{L}_f$ ($\mathcal{L}_f \subset E_f$) the null subspace of the functional f, by K_f the cone $K \cap E_f$, and by $K \oplus E_f$ and $K \oplus \mathcal{L}_f$, respectively, the algebraic sums of the sets K and E_f, and K and $\mathcal{L}_f$. Suppose that on E_f the functional f is positive ([1]): $f(x) \geqslant 0$, if $x \in K_f$. We ask under what conditions f can be extended from E_f to the whole space E with preservation of linearity and positivity. The answers to this question constitute the content of the paper.

Theorem 1. In order that a linear positive functional $f(K_f \not\subset \mathcal{L}_f)$ can be extended from E_f to the whole space E with preservation of linearity and positivity, it is necessary and sufficient that

$$\overline{K \oplus E_f} \neq \overline{K \oplus \mathcal{L}_f}.$$

Definition. An element $u_0 \in K$ will be called an *almost interior element* of the cone K if, for every linear positive functional $f \in E^*$ ($f \neq 0$), the value $f(u_0) > 0$.

Theorem 2. Suppose that the subspace $E_f \subset E$ contains almost interior elements of the cone K. Then, in order that a linear positive functional $f \neq 0$ can be extended from E_f to the whole space E with preservation of linearity and positivity, it is necessary and sufficient that

$$\overline{K \oplus \mathcal{L}_f} \neq E.$$

2. Below, $E/\mathcal{L}_f$ denotes the quotient space, and $K/\mathcal{L}_f$ the corresponding quotient set.

Theorem 3. Suppose that the subspace E_f contains almost interior elements of the cone K. Then, in order that a linear positive functional $f \neq 0$ can be extended, with preservation of linearity and positivity, from E_f to E, it is necessary and sufficient that

$$E/\mathcal{L}_f \neq \overline{K/\mathcal{L}_f}.$$

3. **Theorem 4.** In order that a linear positive functional $f(K_f \not\subset \mathcal{L}_f)$ can be extended from E_f to the whole space E, it is necessary and sufficient that there exist an element $x_0 \in K_f$ and a number $a_0 > 0$ such that, for all $x \in K$,

$$\rho(x + x_0, \mathcal{L}_f) \geqslant a_0.$$

4. Denote by E_f^- the set of all $x \in E_f$ such that $f(x) \leqslant 0$.

Theorem 5. In order that a linear positive functional f $(K_f \not\subset \mathcal{L}_f)$ can be extended from E_f to the whole space E with preservation of linearity and positivity, it is necessary and sufficient that there exist a number $\beta_0 > 0$ such that for all $x \in E_f^-$

$$\rho(x, K) \geqslant \beta_0 \rho(x, \mathcal{L}_f).$$

5. In this section we shall assume that there exists a fixed number $\delta_0 > 0$ such that for any pair of disjoint elements $x, y \in E$ (^{2}) : $|x| \wedge |y| = 0$ the inequality

$$\|x + y\| \geqslant \delta_0(\|x\| + \|y\|) \tag{1}$$

is satisfied.

Theorem 6. Suppose the following conditions are fulfilled:

a) the reproducing cone K is regular and minihedral (^{3});

b) inequality (1) is satisfied.

Then, in order that a linear positive functional f $(K_f \not\subset \mathcal{L}_f)$ can be extended from E_f to the whole space E with preservation of linearity and positivity, it is necessary and sufficient that there exist fixed elements $x_0 \in K_f$ and a number $\gamma_0 > 0$ such that for every $z \in \mathcal{L}_f$ $(z = z_+ - z_-)$ the inequality

$$\|z_- + (x_0 - x_0 \wedge z_+)\| \geqslant \gamma_0.$$

6. We now give a number of sufficient conditions for extendability of linear positive functionals.

Theorem 7. In order that a linear positive functional f $(K_f \neq \theta)$ can be extended from E_f to E with preservation of linearity and positivity, it is sufficient that there exist a number $\beta_0 > 0$ such that for every $x \in K$

$$\rho(x, \mathcal{L}_f) \geqslant \beta_0 \|x\|.$$

Definition. The distance $\rho(x, \mathcal{L}_f)$ will be called **semimonotone** if there exists a fixed number $m_0 > 0$ such that for every pair $x, y \in K$ and $x \leqslant y$ $(y - x \in K)$ it follows that

$$\rho(x, \mathcal{L}_f) \leqslant m_0 \rho(y, \mathcal{L}_f).$$

Theorem 8. In order that a linear positive functional f $(K_f \not\subset \mathcal{L}_f)$ can be extended from E_f to E with preservation of linearity and positivity, it is sufficient that the distance $\rho(x, \mathcal{L}_f)$ be semimonotone.

7. **Theorem 9.** Suppose the following conditions are fulfilled:

a) the reproducing cone K is normal and minihedral;

b) the functional $f \neq 0$ in E_f is linear and positive;

c) in E_f the cone K is spatial ([4]);

d) for every $y \in K_f$, every $x \in K$ satisfying the condition $x \leqslant y$ belongs to K_f.

Then the functional f admits a linear positive extension from E_f to the whole space E.

Theorem 10. Suppose the following conditions are fulfilled:

a) the reproducing cone K is regular and minihedral;

b) inequality (1) is satisfied;

c) in the subspace E_f there exist an almost interior element x_0 of the cone K and a number $a_0 > 0$ such that for every $z \in \mathcal{L}_f$ $(z = z_+ - z_-)$

$$\|x_0 - x_0 \wedge z_+\| \geqslant a_0.$$

Then the linear positive functional f $(K_f \not\subset \mathcal{L}_f)$ can be extended from E_f to the whole space E with preservation of linearity and positivity.

Theorem 11. Suppose that conditions a) and b) of Theorem 10 are satisfied and that there exist fixed numbers $a, b > 0$ such that for every $z \in \mathcal{L}_f$ $(z = z_+ - z_-)$ the inequalities

$$a\|z\| \leq \|z_+\| \leq b\|z\|,$$

$$a\|z\| \leq \|z_-\| \leq b\|z\|$$

hold.

Then the linear positive functional f $(K_f \not\subset \mathcal{L}_f)$ can be extended from E_f to the whole space with preservation of linearity and positivity.

8. Let us now dwell on the question of the existence of linear positive functionals that vanish on a fixed subspace.

Denote by H the closure of the linear hull $\mathcal{L}(K)$ of the cone K, by E_0 a subspace in the space E, and, finally, by $\mathcal{L}_F$ the null subspace of the functional F.

Theorem 12. Let $E_0 \subset H$. Then, in order that there exist a linear positive functional $F \neq 0$ such that $E_0 \subset \mathcal{L}_F$, $K \not\subset \mathcal{L}_F$, it is necessary and sufficient that

$$\overline{K \oplus E_0} \neq H.$$

Theorem 13. Let the cone K contain almost interior elements. Then, in order that there exist a linear positive functional $F \neq 0$ such that $E_0 \subset \mathcal{L}_F$, it is necessary and sufficient that for every almost interior element $x_0 \in K$ the inequality

$$\inf_{x \in K} \rho(x + x_0, E_0) > 0$$

hold.

Theorem 14. Suppose there exist a fixed number $a_0 > 0$ and an almost interior element $x_0 \in K$ such that for every $y \in E$ one can indicate a functional $f_y \in K^*$ $(\|f_y\| = 1)$ such that

$$f_y(y) = 0, \qquad f_y(x_0) \geq a_0.$$

Then there exists a linear positive functional $F \neq 0$ such that $\mathcal{L}_F \supset E_0$.

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References

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[3] M. A. Krasnosel' skii, *Positive Solutions of Operator Equations*, Moscow, 1962.
[4] I. A. Bakhtin, *Siberian Mathematical Journal*, 6, no. 2, 262 (1965).

Note: Figure translations are in progress. See original paper for figures.