WAVE ADIABATS FOR MEDIA WITH AN ARBITRARY EQUATION OF STATE

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Abstract

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WAVE ADIABATS FOR MEDIA WITH AN ARBITRARY EQUATION OF STATE

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We shall consider a gas (liquid) with anomalous thermodynamic properties. In particular, if $P = P(V, S)$ is the pressure, specified as a function of the specific volume V and entropy S, then we assume that the quantity $(\partial^2 P/\partial V^2)_S$ changes sign, although we shall exclude the possibility of phase transitions. It is known (see, for example, (7)) that, in the decay of a discontinuity in a medium with anomalous properties, there arises a flow pattern containing several shock waves following one another in the same direction and separated by centered waves.

Let us consider the set of initial discontinuities of a gas with a fixed state P_0, V_0, u_0 on one side (u_0 is the velocity). The set of states arising at the point of discontinuity after decay will be called the **wave adiabat with center at the point** (V_0, P_0). As we shall see below, the wave adiabat $W(P_0, V_0)$ consists of segments of the Hugoniot adiabat $H(P_0, V_0)$ with center at the same point (V_0, P_0), segments of the Poisson adiabat, and envelopes of certain families of Hugoniot adiabats.

The shock adiabat is given by the equations (see (5))

$$\varepsilon - \varepsilon_0 = \frac{1}{2}(P + P_0)(V_0 - V), \quad (u - u_0)^2 = (P - P_0)(V_0 - V),$$
$$J^2 = (P - P_0)/(V_0 - V). \tag{1}$$

The properties of shock adiabats have been studied, for example, in works (1, 2). We study the general properties of wave adiabats, in contrast to shock adiabats, for a broader class of media.

For nonconvex equations of state, the condition of entropy increase $S > S_0$ at shock waves proves insufficient for the unambiguous selection of the wave adiabat. Therefore, instead of $S > S_0$, two requirements are postulated on the wave adiabat:

1) it is assumed that at every point on the Hugoniot adiabat the condition

$$-\partial P(V, S)/\partial V \geq (P - P_0)/(V_0 - V) \geq -\partial P(V_0, S_0)/\partial V$$

is satisfied;

2) the wave adiabat $W(P_0, V_0)$ is a continuous curve in the plane of the variables V, P.

The first requirement constitutes the principle of stability of discontinuities in the medium (see (3, 4)).

The portions of the Hugoniot adiabat $H(P_0, V_0)$ belonging to the wave adiabat $W(P_0, V_0)$ will be called **physical portions**.

Let us assume that the general requirements of thermodynamics are satisfied (see (6)):

$$T\,dS = d\varepsilon + P\,dV, \quad T > 0, \quad c_v > 0, \quad (\partial P/\partial V)_S < 0,$$
$$-\left(\frac{\partial P}{\partial V}\right)_S \geq \frac{c_v}{T}\left(\frac{\partial P}{\partial S}\right)_V^2. \tag{2}$$

* In thermodynamics there is also the identity

$$-\left(\frac{\partial P}{\partial V}\right)_S = \frac{\gamma}{\gamma - 1}\frac{c_v}{T}\left(\frac{\partial P}{\partial S}\right)_V^2$$

(the existence of this identity was pointed out to us by G. Ya. Galin after discussion of the work at a seminar). In this connection we note that, in the case when it holds, instead of restriction (3) one may use the following:

$$(P - P_0)(V_0 - V) < \frac{4\gamma}{\gamma - 1} c_{vT},$$

We shall study the wave adiabat under the assumption that, on the physical portions of the Hugoniot adiabat, the inequalities

$$(P - P_0)(V_0 - V) < 4c_{vT} \quad \text{for } V \leq V_0; \tag{3}$$

$$(\partial P/\partial S)_V > -2T/(V - V_0) \quad \text{for } V > V_0 \tag{4}$$

are satisfied.

It is also assumed that all the media under consideration have sufficiently smooth equations of state.

We give, without proof, several theorems on the properties of wave adiabats.

Fig. 1

Theorem 1. Points of the Hugoniot adiabat $H(P_0, V_0)$ at which inequality (3) or (4) is satisfied are not singular; in a neighborhood of these points the curve

Fig. 1. Diagram in the p, V plane with the Hugoniot curve H, points (V_0, P_0), (V_1, P_1), (V_2, P_2), (V_3, P_3), and the Poisson adiabat.

Figure 1: Fig. 1. Diagram in the p, V plane with the Hugoniot curve H, points (V_0, P_0), (V_1, P_1), (V_2, P_2), (V_3, P_3), and the Poisson adiabat.

$H(P_0, V_0)$ exists and its equation is representable in the form of a single-valued function of V or P.

Theorem 2. The portion of the curve $H(P_0, V_0)$ lying to the left (right) of the ray drawn from the center (V_0, P_0) for $V < V_0$ $(V > V_0)$, between the point of tangency (V_1, P_1) (Fig. 1)* and the next point of intersection (V_3, P_3), is not physical.

Theorem 3. The entropy increases monotonically on the physical portions of $H(P_0, V_0)$: 1) with increasing pressure for $V \le V_0$; 2) with decreasing pressure for $V \ge V_0$.

Theorem 4. For the velocity to increase monotonically with pressure on the physical portions of $H(P_0, V_0)$, it is sufficient that inequalities (3)–(4) be satisfied.

It follows from this theorem that $dP/dV > J^2$ on the segment (a, b) (Fig. 2), which, as is easily shown, is physically admissible (d/dV is the derivative with respect to V along $W(P_0, V_0)$).

Let R denote the curve whose points satisfy the system of equations

$$\varepsilon - \tilde{\varepsilon} = \frac{1}{2}(P + \tilde{P})(\tilde{V} - V), \qquad (u - \tilde{u})^2 = (P - \tilde{P})(\tilde{V} - V),$$

$$J^2 = \frac{P - \tilde{P}}{\tilde{V} - V}, \qquad J^2 = \tilde{J}^2,$$

* Fig. 1 is drawn for the case $(\partial^2 P/\partial V^2)_S > 0$ at the point (V_0, P_0).

where $\varepsilon, \widetilde{P}, \tilde{u}, \widetilde{J}$ are functions of $\widetilde{V}$ for $\widetilde{S} =$ const, satisfying the relations $\widetilde{P} = P(\widetilde{V})$, $d\varepsilon = -\widetilde{P}\, d\widetilde{V}$, $(d\tilde{u})^2 = -d\widetilde{P}\, d\widetilde{V}$, $\widetilde{J}^2 = -d\widetilde{P}/d\widetilde{V}$.

Similar curves were first introduced in [8].

Lemma. *The curve R is the envelope of the family of shock adiabats $H(\widetilde{P}, \widetilde{V})$, whose centers are located on the segment of the Poisson adiabat for $V_2 \le \widetilde{V} \le V_1$ (Fig. 1).*

We now turn to the construction of the wave adiabat $W(P_0, V_0)$. It is easy to show that at the point (V_1, P_1) the equalities

$$(\partial P/\partial V)_S = (P - P_0)(V - V_0), \qquad dS/dV = 0,$$

$$dP/dV = (P - P_0)/(V - V_0),$$

hold simultaneously, and the sign of the derivatives d^{nP}/dV^n, d^{nS}/dV^n coincides with the sign of $(\partial^{nP}/\partial V^n)_S$, provided that $(\partial^{kP}/\partial V^k)_S = 0$ for $2 \leq k \leq n-1$.

Taking this into account, we follow the Poisson adiabat, on which $(\partial^2 P/\partial V^2)_S \leq 0$, up to the inflection point (V_2, P_2). Further, from the point (V_2, P_2) we draw the curve R, the existence and uniqueness of which are easy to prove. The properties of the curve R are analogous to the properties of the curve $H(P_0, V_0)$. It is noteworthy that on R the entropy S and the velocity u also increase monotonically with increasing pressure P. We note that from [2] it follows that the curve R passes through the point (V_3, P_3) of intersection of $H(P_0, V_0)$ with the ray.

Fig. 2

Fig. 3

Combining the physical portions of the Hugoniot adiabat $H(P_0, V_0)$, segments of the Poisson adiabats $S = \text{const}$, and the curves R connecting them, we obtain a continuous curve $W(P_0, V_0)$, which is the wave adiabat.

We note that the curves R of the adiabat $W(P_0, V_0)$ themselves may have non-physical portions, which are determined analogously to the preceding case. In this case they are also replaced by segments of Poisson adiabats and by new envelopes R'. We do not describe this construction procedure in detail here.

The presence of a segment with a curve R on the wave adiabat $W(P_0, V_0)$ means that for $P_3 < P < P_2$ there exist two or more shock waves separated by continuous compression waves.

Everything that has been said here for compression waves is also valid, with some modification, for rarefaction waves. We note one small difference arising from the difference between estimates (3) and (4) for compression and rarefaction. At the point a (Fig. 2), where there is a vertical tangent to $H(P_0, V_0)$, the equalities

$$(\partial P/\partial S)_V = 2T/(V_0 - V), \qquad dV/dS = 0, \qquad dV/dP = 0.$$

will hold simultaneously.

In view of (4), for $V \geq V_0$ the curve $H(P_0, V_0)$ has no vertical tangent. Thus, a segment analogous to the segment (a, b) does not exist for rarefaction waves.

Remark 1. In order to prove the uniqueness of the solution of the classical problem of the decay of an arbitrary discontinuity (the Riemann problem; see [4] or [5]), it is necessary that the velocity vary monotonically with pressure along the wave adiabat. From Theorem 4 and the analogous properties of the curves R it follows that, under the imposed restrictions, this will be satisfied and, thus, the decay problem has no more than one solution.

Remark 2. Inequalities (3)–(4) are always satisfied in some sufficiently small neighborhood of the initial point (V_0, P_0), independently of the thermodynamic properties of the medium. Therefore, in this finite region we have proved the existence and uniqueness of the wave adiabat $W(P_0, V_0)$.

Remark 3. Estimate (3) follows from the inequalities

$$-\frac{2T}{V_0 - V} < \left(\frac{\partial P}{\partial S}\right)_V < \frac{2T}{V_0 - V},$$

which hold on the segment $(a, a_0]$ of the adiabat $H(P_0, V_0)$ (Fig. 2); however, they are not satisfied on the segment $[a, b]$, since in this case

$$\left(\frac{\partial P}{\partial S}\right)_V \geq \frac{2T}{V_0 - V},$$

and therefore here inequality (3) is an additional restriction.

One can give another equivalent form of inequality (3):

$$\oint_c T\, dS < 4c_v T,$$

where the integration is over the contour indicated in Fig. 3.

As noted above, inequality (3) is always satisfied in some neighborhood of the point (V_0, P_0). Let us consider this inequality for $V \ll V_0$. Here one can distinguish two regions: a) $V \geq V_{0k}$ and b) $V < V_{0k}$, where V_{0k} is the specific volume of the medium at $T = 0$, $P = 0$ (see (7)) (for ideal media we put $V_{0k} = 0$). It turns out that for $V \geq V_{0k}$ inequality (3) follows from the cruder condition

$$\int_0^T c_v\big|_{v=\text{const}}\, dT < 2aT.$$

Thus, checking the validity of inequality (3) in the region $V \geq V_{0k}$ reduces to investigating the behavior of the heat capacity c_v with temperature (the heat-capacity problem).

As for inequality (4), here it is clear that as $V \to \infty$ or $T \to 0$ it passes into

$$\left(\frac{\partial P}{\partial S}\right)_V > 0.$$

Remark 4. It is easy to prove the following facts:

1) At the point (V_3, P_3) (Fig. 1) a break of the wave adiabat always occurs if at it $dS \neq 0$.
2) For media in which the energy is a function with variables separable in V, S ($\varepsilon = \Phi(V)\varphi(S)$), the turning point a of the Hugoniot shock adiabats (Fig. 2) does not exist.

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Note: Figure translations are in progress. See original paper for figures.