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Abstract

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MATHEMATICS

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INVESTIGATION OF A WEAK SOLUTION OF A MIXED PROBLEM FOR A HYPER- BOLIC EQUATION WITH A NONLINEAR PART

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The main question studied in this note is the question of the correctness and existence of a certain kind of solutions of the mixed problem for the hyperbolic equation

$$u_{tt} = \mathcal{L}u + F(t, x, u) \quad (1)$$

in the cylinder $\Omega_T = [0, T] \times g$,

$$u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x); \quad (2)$$

$$B_\Gamma u = 0; \quad (3)$$

here g is an N -dimensional domain; Γ is the boundary of this domain; $x = (x_1, x_2, \dots, x_N)$; $\varphi(x), \psi(x)$ are functions prescribed in the domain g ; $F(t, x, u)$ is a prescribed function defined in the strip $\Omega_T \times [-l, l]$, where $0 < l \leq +\infty$. \mathcal{L} is a self-adjoint differential expression which, together with condition (3), generates a self-adjoint positive operator; denote by D_L the domain of definition of this operator.

Mixed problems for quasilinear equations have been studied by L. Lichtenstein, M. R. Siddiqi, D. Levi, M. Krzyżański, I. Schauder, W. Barbăuti, A. I. Guseinov, K. I. Khudaverdiev, K. K. Gasanov, V. N. Gol'dberg, Yu. N. Neimark, S. A. Zhautykov, A. Sh. Guseinov, G. N. Khalilov, and others. In contrast to the works of these authors, we study a weak (in a certain sense) solution of problem (1), (2), (3).

It is clear that the operator \mathcal{L} is invertible. Let some iteration of the inverse operator \mathcal{L}^{-1} be a completely continuous operator in $\mathcal{L}_2(g)$. Obviously, \mathcal{L} has

a complete orthonormal system of eigenfunctions $\{v_n(x)\}$. Denote by $\{\lambda_n^2\}$ the system of eigenvalues corresponding to $\{v_n(x)\}$; by $W_2(\mathcal{L})$ the set of functions $u(t, x)$ which, for any $t \in [0, T]$, are elements of D_L for all $x \in g$ and have continuous second derivatives with respect to t , with $u(t, x) \equiv 0$ if $t \in (T - \delta, T)$, where $0 < \delta < T$; by $\mathcal{L}_2(\Omega_T)$ the set of measurable functions $a(t, x)$ for which

$$\int_0^T \int_g a^2(t, x) dx dt < +\infty;$$

and by B_α (where $\alpha = (a, 2, m)$) the set of functions from $\mathcal{L}_2(\Omega_T)$ for which the Fourier coefficients with respect to the eigenfunctions of the operator \mathcal{L} satisfy the inequality

$$\sum_{n=1}^{\infty} \left[\sum_{k=0}^m \lambda_n^{\alpha_1} \left(\max_{0 \leq t \leq T} |A_n^{(k)}(t)| \right) \right]^2 < +\infty.$$

A function $F(t, x, u)$ of $(N + 2)$ variables, $t \in [0, T]$, $x \in g$, $-l \leq u \leq l$, is said to satisfy the Carathéodory conditions if F is continuous in u for almost all $(t, x) \in \Omega_T$ and measurable in (t, x) for every $u \in [-l, l]$. We shall assume that $F(t, x, u)$ satisfies the Carathéodory conditions.

If $u(t, x) \in \mathcal{L}_2(\Omega_T)$ satisfies the integral identity

$$\int_0^T \int_g \{u[\Phi_{tt} - \mathcal{L}\Phi] - F(t, x, u)\Phi(t, x)\} dx dt + \int_g \left[\psi(x) \frac{\partial \Phi}{\partial t} - \varphi(x)\Phi \right]_{t=0} dx = 0$$

for every $\Phi(t, x) \in W_2(\mathcal{L})$, then the function $u(t, x)$, following S. L. Sobolev, will be called an $(\mathcal{L}_2, W_2(\mathcal{L}))$ -weak solution of problem (1), (2), (3) (briefly, an $(\mathcal{L}_2, W_2(\mathcal{L}))$ -w.s.).

In studying the correctness and existence of an $(\mathcal{L}_2, W_2(\mathcal{L}))$ -w.s. we use an inequality of Gronwall type, contained in Lemma 1, and the continuity of the operator F .

Lemma 1. *Let E be a partially ordered Banach space, F a nondecreasing continuous operator in E , and $u \leq F(u)$, $v_{n+1} = F(v_n)$, where $v_0 = u$. Then, if $v_n \rightarrow v$ as $n \rightarrow +\infty$, then $u \leq v$.*

From this lemma there follows the following assertion. If a nonnegative function $a(t) \in C_{(0, T)}$ satisfies the inequality

$$a(t) \leq C + \int_0^t [b(\tau)a(\tau) + m(\tau)a^\alpha(\tau)] d\tau,$$

where $C \geq 0$ is a constant, $\alpha \in [0, 1]$, and the functions $b(t)$, $m(t)$ are nonnegative and integrable on $(0, T)$, then

$$a(t) \leq \left\{ C^{1-\alpha} \exp \left[(1-\alpha) \int_0^t m(\tau) d\tau \right] + (1-\alpha) \int_0^t b(\tau) \exp \left[(1-\alpha) \int_\tau^t m(r) dr \right] d\tau \right\}^{1/(1-\alpha)}.$$

Theorem 1. *Let the Carathéodory condition hold and let $l = +\infty$. Then, if*

$$|F(t, x, u)| \leq a(t, x) + b(x)|u|,$$

where $a(t, x) \in \mathcal{L}_2(\Omega_T)$, $b(t) \in \mathcal{L}_2(0, T)$, then F acts from $B_{(0,2,0)}(T)$ into $\mathcal{L}_2(\Omega_T)$, is continuous and bounded.

Lemma 2. *If: a) $F(t, x, u)$ satisfies the conditions of Theorem 1; b) $u(t, x)$ is an $(\mathcal{L}_2, W_2(\mathcal{L}))$ -w.s.; c)*

$$A_n(t) = \int_g u(t, x) v_n(x) dx, \quad n = 1, 2, \dots,$$

then the Fourier coefficients $A_n(t)$ satisfy the countable system of nonlinear integral equations

$$A_n(t) = \varphi_n \cos \lambda_n t + \frac{\psi_n \sin \lambda_n t}{\lambda_n} + \frac{1}{\lambda_n} \int_0^t \int_g F \left[\tau, x, \sum_{k=1}^{\infty} A_k(\tau) v_k(x) \right] v_n(x) \sin \lambda_n(t - \tau) dx d\tau, \quad (\text{A})$$

where

$$\varphi_n = \int_g \varphi(x) v_n(x) dx, \quad \psi_n = \int_g \psi(x) v_n(x) dx, \quad n = 1, 2, \dots$$

Using system (A), one can prove the following theorems.

Theorem 2. *Let: a) $\varphi_i(x)$, $\psi_i(x)$, $i = 1, 2$, be elements of the space $\mathcal{L}_2(g)$; b) $F_i(t, x, 0) \in \mathcal{L}_2(\Omega_T)$, and for any u, v from $(-\infty, \infty)$ the inequality*

$$|F_1(t, x, u) - F_2(t, x, v)| \leq k(t)|u - v| + C(t, x),$$

holds, where $C(t, x) \in \mathcal{L}_2(\Omega_T)$, $k(t) \geq 0$ and is integrable on $(0, T)$; c) $u_i(t, x)$ are $(\mathcal{L}_2, W_2(\mathcal{L}))$ -w.s., i.e.

$$\int_0^T \int_g \{u_i[\Phi_{tt} - \mathcal{L}\Phi] - F_i(t, x, u_i)\Phi\} dx dt + \int_g \left[\psi_i \frac{\partial \Phi}{\partial t} - \varphi_i \Phi \right]_{t=0} dx = 0,$$

$$i = 1, 2. \tag{4}$$

Then, if $A_n^{(i)}(t) = \int_g u_i(t, x) u_n(x) dx$, $i = 1, 2$; $n = 1, 2, \dots$, the inequality

$$\sum_{n=1}^{\infty} \max_{0 \leq t \leq T} |A_n^{(2)}(t) - A_n^{(1)}(t)| \leq 3 \left\{ \int_g [\varphi_2(x) - \varphi_1(x)]^2 dx + \frac{1}{\lambda_1} \int_g [\psi_2(x) - \psi_1(x)]^2 dx + \int_0^T \int_g C^2(t, x) dx dt \right\} \\ + \int_0^T \left\{ k(s) \left[\int_0^s \int_g C^2(t, x) dx dt \right] \left(\exp \int_s^T k(r) dr \right) \right\} ds.$$

Theorem 3. Suppose: a) the sequence of eigenfunctions of the operator \mathcal{L} is uniformly bounded, i.e. there exists a number M such that $|v_n(x)| \leq M$ for every n ; b) $u_i(t, x)$ satisfy the integral identity (4), i.e. are $(\mathcal{L}_2, W_2(\mathcal{L}))$ -w.s.; c) in the Carathéodory condition $l = +\infty$,

$$\int_0^T \left[\int_g |F_i(t, x, u_i(t, x))|^2 dx \right]^{q/p} dt < +\infty, \quad i = 1, 2,$$

where $1/p + 1/q = 1$, $p > 1$ is such that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^p} < +\infty;$$

d)

$$|F_2(t, x, u_2(t, x)) - F_1(t, x, u_1(t, x))| \leq a_0(t, x)|u_2 - u_1| + a_1(t, x)|u_2 - u_1|^\alpha + a_2(t, x),$$

where $0 \leq \alpha \leq 1$,

$$\int_0^T \left[\int_g a_i^p(t, x) dx \right]^{q/p} dt < +\infty, \quad i = 0, 1, 2.$$

Then, if

$$A_n^{(i)}(t) = \int_g u_i(t, x) v_n(x) dx, \quad b_n^{(i)}(t) = \varphi_n^{(i)} \cos \lambda_n t + \psi_n^{(i)} \frac{\sin \lambda_n t}{\lambda_n},$$

where

$$\varphi_n^{(i)} = \int_g \varphi_i(x) v_n(x) dx, \quad \psi_n^{(i)} = \int_g \psi_i(x) v_n(x) dx, \quad i = 1, 2; n = 1, 2, \dots,$$

the inequalities

$$\begin{aligned} \sum_{n=1}^{\infty} \max_{0 \leq t \leq T} |A_n^{(2)}(t) - A_n^{(1)}(t)| \leq & \left\{ \left[\alpha_0 \left(\sum_{n=1}^{\infty} \max_{0 \leq t \leq T} |b_n^{(2)}(t) - b_n^{(1)}(t)| + \alpha_1 \int_0^T \left(\int_g a_2^p(t, x) dx \right)^{q/p} dt \right) \right]^{1-\alpha} \right. \\ & \cdot \exp(1 - \alpha) \alpha_2 \int_0^T \left(\int_g a_1^p(t, x) dx \right)^{q/p} dt + (1 - \alpha) \int_0^T \left(\int_0^\tau \int_g a_2(\xi, x) dx d\xi \right) \exp \int_\tau^T \left[\int_g a_0^p(\eta, x) dx \right]^{q/p} d\eta \end{aligned}$$

where $\alpha_0, \alpha_2, \alpha_1$ are constants.

Theorem 4. Suppose: a) in the Carathéodory condition $l = +\infty$; b) $F(t, x, u(t, x)) \in \mathcal{L}_2(\Omega_T)$, if $u(t, x)$ is an $(\mathcal{L}_2, W_2(\mathcal{L}))$ -w.s.; c)

$$|F(t, x, u) - F(t, x, v)| \leq \frac{|u - v|}{t}, \quad \lambda_1 \geq 1.$$

Then problem (1), (2), (3) cannot have more than one $(\mathcal{L}_2, W_2(\mathcal{L}))$ -w.s.

It is not difficult to show that if: a) the boundary of the domain g is such that Green's formula is applicable; b) $\varphi(x), \psi(x)$ are elements of $\mathcal{L}_2(g)$; c) $|F(t, x, u)| \leq a(t, x) + b|u|$ for all $u \in (-\infty, \infty)$, where $a(t, x) \in \mathcal{L}_2(\Omega_T)$, b is a constant, and the sequence $\{A_n(t)\}$ is a solution of system (A), then the function

$$u(t, x) = \sum_{n=1}^{\infty} A_n(t) v_n(x)$$

is an $(\mathcal{L}_2, W_2(\mathcal{L}))$ -w.s. (here $v_n(x)$ are eigenfunctions of the operator \mathcal{L}).

Theorem 5. Suppose: a) in the Carathéodory condition $l = +\infty$; b) $\varphi(x), \psi(x) \in \mathcal{L}_2(g)$; c)

$$|F(t, x, u)| \leq a(t, x) + a_1(t, x)|u|^\alpha + b(t)|u|;$$

here $a(t, x) \in \mathcal{L}_2(\Omega_T)$, $b(t) \in \mathcal{L}_2(0, T)$, $b(t) > 0$,

$$\int_0^T \left[\int_g a_1^{2/(1-\alpha)}(t, x) dx \right]^{1-\alpha} dt < +\infty.$$

Then problem (1), (2), (3) has at least one $(\mathcal{L}_2, W_2(\mathcal{L}))$ -s.r., and there exist some positive numbers T, R such that

$$\sum_{n=1}^{\infty} \max_{0 \leq t \leq T} |A_n(t)|^2 \leq R^2,$$

where $A_n(t)$ are the Fourier coefficients of the $(\mathcal{L}_2, W_2(\mathcal{L}))$ -s.r. with respect to the eigenfunctions of the operator \mathcal{L} .

Theorem 6. Suppose: a) the set of eigenvalues of the operator \mathcal{L} is bounded by some number M , i.e. $|v_n(x)| \leq M$ for any n ;

b) $|F(t, x, u)| \leq a(t, x)$ for all $u \in (-a, a)$, where

$$\int_0^T \left[\int_g a^p(t, x) dx \right]^{q/p} dt < +\infty$$

and the number p is such that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^p} < +\infty,$$

q is conjugate to p ;

c) $\varphi(x), \psi(x), T$ are such that

$$\sum_{p=1}^{\infty} \left(|\varphi_n| + \frac{1}{\lambda_n} |\psi_n| \right) \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^p} \right)^{1/p} M_1 \left\{ \int_0^T \left[\int_g a^p(t, x) dx \right]^{1/p} dt \right\}^{1/q} < l;$$

here φ_n, ψ_n are the Fourier coefficients of the functions $\varphi(x), \psi(x)$ in the expansion with respect to the eigenfunctions of the operator \mathcal{L} .

Then problem (1), (2), (3) has at least one $(\mathcal{L}_2, W_2(\mathcal{L}))$ -s.r., and

$$\sum_{n=1}^{\infty} \max_{0 \leq t \leq T} |A_n(t)| < l.$$

Remark. Under the conditions of Theorem 6, the $(\mathcal{L}_2, W_2(\mathcal{L}))$ -s.r. is a discontinuous function.

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Note: Figure translations are in progress. See original paper for figures.

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