

# ON THE PROBLEM OF MINIMALITY FOR SUBGROUPS IN LOCALLY FINITE GROUPS

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## Abstract

## Full Text

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MATHEMATICS

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# ON THE PROBLEM OF MINIMALITY FOR SUBGROUPS IN LOCALLY FINITE GROUPS

(Presented by Academician V. M. Glushkov, 13 XI 1967)

1. In the present article the author continues his investigations begun in <sup>(1-4)</sup>. The problem of minimality for subgroups in locally finite graduses <sup>(5,6)</sup> is studied more intensively. In this case the problem of minimality has been reduced to a very concrete situation (see Theorem 2).

In § 3 a theorem from <sup>(4)</sup> is refined; its proof was given in full at the Batumi Symposium in October 1966, for a group of regular cardinality <sup>(7)</sup>, i.e. a cardinality not representable as a sum of smaller cardinalities with a smaller cardinality of the set of summands. From this theorem there follows a corollary pertaining to the problem of countability for groups with the minimality condition for subgroups <sup>(5)</sup>.

In § 4, Theorem 6 of <sup>(1)</sup> is generalized to periodic groups of even order.

2. **Definition 1.** A group in which every proper subgroup is extremal will be called **quasi-extremal**.

**Definition 2.** A Sylow 2-subgroup of the unitary group  $U_3(4)$  will be called a **Suzuki 2-group of order  $2^6$  of type 1**.

**Lemma.** *It is sufficient to solve the minimality problem for a group, assuming it to be infinite, simple, quasi-extremal; moreover, if the group is locally finite, then it contains involutions.*

The lemma follows from the minimality condition for subgroups, the theorem of S. N. Chernikov <sup>(8)</sup>, and the solvability of a finite group of odd order <sup>(9)</sup>.

**Theorem 1.** *Let  $G$  be an infinite simple locally finite quasi-extremal group;  $S$  a Sylow 2-subgroup of the group  $G$ ;  $Q$  the subgroup generated by all involutions from  $S$ .*

*Then one of the following two assertions 1 and 2 holds:*

1. *The subgroup  $S$  is one of the groups:*

$$\alpha) \quad a^{2^n} = t^2 = 1, \quad tat = a^{-1+2^{n-1}}, \quad n > 2;$$

$\beta)$  a Suzuki 2-group of order  $2^6$  of type 1;

$$\gamma) \quad a^{2^n} = b^{2^n} = t^2 = 1, \quad ab = ba, \quad tat = a^{-1}, \quad tbt = ba^{-1}, \quad n \geq 2.$$

The centralizer of an involution from  $G$  is a maximal subgroup in  $G$ .

2. a) The subgroup  $S$  is finite;

$\beta)$  the subgroup  $H = N_G(Q)$  is maximal in  $G$  and the factor group  $H/C_G(Q)$  is nonsolvable;

$$\gamma) \quad H = \{r\} \cdot C_G(i),$$

where  $i$  is some involution from  $S$ ;  $\{r\}$  is a cyclic Hall subgroup in  $G$ , inducing in  $Q$  a group of automorphisms acting transitively on the set of involutions of  $Q$ , and the element  $r$  is strictly real with respect to some involution of  $G$ ;

$$\delta) \quad N_G(\{r\}) = D\lambda\{t\},$$

where  $\{t\}$  is a cyclic 2-subgroup,  $D$  a finite subgroup of odd order from  $H$ ;

$\lambda)$  if  $H_1$  is a subgroup of  $G$  conjugate to  $H$ , and  $H_1 \neq H$ , then  $D_1 = H \cap H_1$  is conjugate in  $D$ .

In view of the Feit-Thompson theorem <sup>(9)</sup>, the following follow from Theorem 1:

**Corollary 1.** If, in a locally finite group with the minimality condition for subgroups, the Sylow 2-subgroup is Abelian, then the group is extremal.

**Corollary 2.** A locally finite group with Abelian Sylow primary subgroups is a direct product of a finite number of quasicyclic groups if and only if it satisfies the minimality condition for subgroups and has no subgroups of finite index.

3. **Theorem 2.** Let  $G$  be an infinite periodic group of regular cardinality, and let  $k$  be an involution in  $G$ . Then either the centralizer of some involution in  $G$  has the cardinality of the group  $G$ , or in  $G$ , for every involution, there exists a subgroup of cardinality  $G$  with nontrivial center, not containing this involution.

**Proof.** If  $M$  is some set, then by  $|M|$  we shall denote the cardinality of the set  $M$ .

Let  $|C_G(k)| < |G|$ . Then, in view of the conditions of the theorem, the class of elements conjugate to  $k$  in  $G$  has the cardinality of the group  $G$ .

Consider elements of the form  $kg^{-1}kg$ ,  $g \in G$ . Denote the set of such elements by  $L$ . The set  $L$  can be represented as the sum of two disjoint subsets  $L = T + N$ , where  $T$  contains all elements of odd order, and  $N$  those of even order.

Since  $|L| = |G|$ , and the cardinality  $|G|$  cannot be represented as a sum of smaller cardinalities and with summands of smaller cardinality, one of  $T$  and  $N$  has the cardinality of the group  $G$ . If  $|N| = |G|$ , then it is not difficult to show that the cardinality of the centralizer of some involution in  $G$  coincides with  $|G|$ .

Let  $|T| = |G|$ . Applying to the set  $T$  the same method of reasoning as in the proof of the theorem from <sup>(4)</sup>, we shall prove the existence of a subgroup with nontrivial center of cardinality  $|G|$ , not containing the involution  $k$ . The theorem is proved.

**Corollary 1.** Let all proper subgroups in a periodic group  $G$  be countable. Then one of the following assertions holds:

- 1) the group  $G$  is of odd order;
- 2) the Sylow 2-subgroup is invariant in  $G$ ;
- 3) the group  $G$  is countable.

**Corollary 2.** To solve the countability problem for groups with the minimality condition for subgroups, it suffices to solve it for periodic groups of odd order.

4. In accordance with <sup>(1)</sup>, a group in which every proper infinite subgroup is locally nilpotent will be called an  $\bar{S}$ -group.

**Theorem 3.** Let  $G$  be a periodic centerless  $\bar{S}$ -group. Then one of the following assertions holds:

- 1) the group  $G$  is of odd order;
- 2) the Sylow 2-subgroup of  $G$  is either cyclic (quasicyclic), or a generalized (infinite) quaternion group. The centralizer of any involution in  $G$  is infinite and is mutually prime to its conjugates;
- 3) the centralizer of any involution is an infinite 2-group;
- 4) the group  $G$  is either locally nilpotent or extremal.

**Corollary.** If a periodic  $\bar{S}$ -group has an almost regular involution, then it is extremal.

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*Note: Figure translations are in progress. See original paper for figures.*

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