ON A BOUNDARY VALUE PROBLEM FOR LINEAR PARABOLIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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Abstract

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ON A BOUNDARY VALUE PROBLEM FOR LINEAR PARABOLIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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In the present article the first boundary value problem is considered for a linear equation of parabolic type

$$Lu \equiv \frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i}\left(a_{ij}u_{x_j} + a_i u + f_i\right) + b_i u_{x_i} + au + f = 0 \tag{1}$$

with unbounded lower-order coefficients and free terms in the cylinder $Q_T = D \times [0 \le t \le T]$ $(T < \infty)$, whose base is a bounded domain D of the Euclidean space E^n.

We take the initial and boundary conditions in the form

$$u|_{t=0} = \psi_0(x), \qquad u|_{S_T} = 0. \tag{2}$$

Numerous works have been devoted to the study of existence and uniqueness of the generalized solution of problem (1), (2) in one class or another (see the detailed bibliography in ([1–5])). In the monograph ([1]) the latest results in this direction are set forth in detail. In the case when a_i^2, b_i^2, a with respect to the variable x belong to L_p, $p > n$, in ([1]) conditions are given ensuring uniqueness and existence of a generalized solution from V_2 of problem (1), (2); here the usual embedding theorems are used essentially.

However, in the case $p < n$ these conditions are insufficient for the existence and uniqueness of a generalized solution from V_2 of problem (1), (2). In this case, some additional conditions are needed for unique solvability. Moreover, in the case $p < n$ the usual theorems do not work.

In the present article, in the case $1 < p < n$, in terms of spaces with mixed norm, conditions on the data of the problem are found under which the unique solvability of problem (1), (2) in V_2 holds. In addition, the smoothness of the solution $u(x,t)$ with respect to t is studied (for the definitions of the spaces V_2, $V_2^{1,0}$, V_2^*, see ([1])).

Let s be a fixed natural number not exceeding n; E^n is the n-dimensional Euclidean space of points $\bar{x}$; E^{n+1} is the $(n+1)$-dimensional Euclidean space of points $(\bar{x},t)$; $E^s(E^{n-s})$ is the s-dimensional ($(n-s)$-dimensional) space of points $\bar{x}_s(\bar{x}_{n-s})$, where $\bar{x}=(\bar{x}_s,\bar{x}_{n-s})$, $\bar{x}_s=(x_1,x_2,\dots,x_s)$, $\bar{x}_{n-s}=(x_{s+1},x_{s+1},\dots,x_n)$; D is a bounded domain of class $C(\bar{h},\omega)$ in E^n (for the definition of the class $C(\bar{h},\omega)$, see ([2])); Q_T is the cylinder $D\times[0\le t\le T]$; $D_1=D\cap(\bar{x}_{n-s}=\text{const})$; $D_2=\operatorname{pr}_{E^{n-s}}D$; $L_{(p_1,p_2,l)}(Q_T)$ is the Banach space consisting of all functions measurable on Q_T and having finite norm

$$\|f\|_{L_{(p_1,p_2,l)}(Q_T)}=\big\|\|f\|_{L_{(p_1,p_2)}(D)}\big\|_{L_l(0\le t\le T)}.$$

Let $\Omega^{(\varepsilon)}$ denote the following set of points (r_1,r_2) of the plane r_1Or_2:

$$\Omega_k^{(\varepsilon)}\equiv\left\{\begin{array}{l}(r_1,r_2):\quad kr_1r_2-(n-s)r_1-sr_2-\varepsilon=0;\\ \infty>r_1>\begin{cases}1, & \text{if } s<k,\\ s/k, & \text{if } s\ge k;\end{cases}\\ \infty<r_2>\begin{cases}1, & \text{if } n-s<k,\\ (n-s)/k, & \text{if } n-s\ge k;\end{cases}\\ r_1\ge r_2.\end{array}\right\}.$$

Put, for any real positive number $p\ge 1$,

$$\Omega_{k;p}^{(\varepsilon)}\equiv\Omega_k^{(\varepsilon)}\cap\{r_1,r_2\ge p\}.$$

Obviously, $\Omega_{k;1}^{(\varepsilon)}=\Omega_k^{(\varepsilon)}$. Put $\Omega_{k;p}^{(0)}=\Omega_{k;p}$. The class $X_{(m-2/l_1)}$ of Banach spaces $L_{(r_1,r_2,l)}(Q_T)$ is defined by the equality

$$X_{(m-2/l_1)}\equiv\{L_{(r_1,r_2,l_1)}(Q_T);\ (r_1,r_2)\in\Omega_{(m-2/l_1)},\ l_1>1\}.$$

We define the class $\widehat{X}_{(2-2/l_1);p}$ by the equality

$$\widehat{X}_{(2-2/l_1);p}=\bigcup_{(r_1,r_2)\in\Omega_{(2-2/l_1);p}}L_{(r_1,r_2,l_1)}(Q_T).$$

Let

$$R_{ij}=R_{ij}(r_{ij1},r_{ij2},l_1),\qquad P_{ij}=P_{ij}(p_{ij1},p_{ij2},l),\quad i=1,2,\dots,n;\quad j=1,2,$$

where $p_{ijk} = 2r_{ijk}$, $k = 1, 2$; $l = 2l_1$,

$$R'_{ij} = R'_{ij}(r'_{ij1}, r'_{ij2}, l'_1), \qquad P'_{ij} = P'_{ij}(p'_{ij1}, p'_{ij2}, l'),$$

$$R_3 = R_3(r_{31}, r_{32}, l_1), \qquad R_4 = R_4(r_{41}, r_{42}, l'),$$

$$P_3 = P_3(p_{31}, p_{32}, l), \qquad P_4 = P_4(p_{41}, p_{42}, l), \qquad p_{kj} = 2r_{kj}.$$

Theorem 1. If $u(x,t) \in \dot{V}_2(Q_T)$ and the numbers q_1, q_2, l satisfy the conditions

$$1/l + s/2q_1 + (n-s)/2q_2 = n/4, \qquad 2 \le q_1, q_2, l,$$

then

$$\|u\|_{L(q_1,q_2,l)(Q_T)} \le c \ \operatorname*{vrai\,max}_{0 \le t \le T} \|u\|^{1-2/l}_{L_2(D)} \|u_x\|^{2/l}_{L_2(Q_T)}.$$

Corollary. Under the condition of Theorem 1, the estimate

$$\|u\|_{L(q_1,q_2,l)(Q_T)} \le c\,|u|_{Q_T}$$

holds.

Lemma 1. If the numbers q_1, q_2, l satisfy the conditions

$$1/l + s/2q_1 + (n-s)/2q_2 > n/4, \tag{3}$$

then one can choose numbers $\tilde{q}_1, \tilde{q}_2, \tilde{l}$ satisfying the conditions

$$1/\tilde{l} + s/2\tilde{q}_1 + (n-s)/2\tilde{q}_2 = n/4, \qquad \tilde{l}/\tilde{q}_2 = l/q_2,$$

$$1/q_1 - 1/q_2 = 1/\tilde{q}_1 - 1/\tilde{q}_2. \tag{4}$$

Lemma 2. If the numbers q_1, q_2, l and $\tilde{q}_1, \tilde{q}_2, \tilde{l}$ satisfy conditions (3), (4) and $f \in L_{(\tilde{q}_1,\tilde{q}_2,\tilde{l})}(Q_T)$, then the estimate

$$\|f\|_{L(q_1,q_2,l)(Q_T)} \le \|f\|_{L(\tilde{q}_1,\tilde{q}_2,\tilde{l})(Q_T)} \left\{ \int_0^T [\text{mes}\, A(t)]^{\tilde{l}/q}\, dt \right\}^{(\tilde{l}-l)/\tilde{l}l};$$

$A(t)$ is the set of points $x \in D$ where $|f(x,t)| > 0$.

Consider equation (1). Suppose the coefficients satisfy the conditions

$$\nu\xi_i\xi_i \le a_{ij}\xi_i\xi_j \le \mu\xi_i\xi_i, \qquad \nu,\mu = \text{const} > 0, \qquad a_{ij} = a_{ji}; \tag{5}$$

$$\|a_i^2\|_{L_{R_{i1}}(Q_T)}, \quad \|b_i^2\|_{L_{R_{i2}}(Q_T)}, \quad \|a\|_{L_{R_3}(Q_T)} \le \mu_1; \tag{6}$$

$$\left\|\left(\sum f_i^2\right)^{1/2}\right\|_{L_2(Q_T)}, \quad \|f\|_{L_{R_4}(Q_T)} \le \mu_1, \tag{7}$$

where

$$L_{R_3}(Q_T),\ L_{R_{ij}}(Q_T) \in X_{(2-2/l_1)}, \qquad i = 1, 2, \dots, n; \quad j = 1, 2; \tag{8}$$

$$L_{R_4} \in X_{((n+4)/2-2/l')}. \tag{9}$$

Denote

$$L_1(u,\eta) \equiv \int_D [(a_{ij}u_{x_j} + a_i u)\eta_{x_i} + (b_i u_{x_i} + au)\eta]\, dx,$$

$$L_2(\hat{f},\eta) \equiv \int_D (f\eta + f_i\eta_{x_i})\, dx.$$

Theorem 2. Suppose $u(x,t) \in \mathring{V}_2(Q_T)$, and u satisfies, for almost all t_1 and t_2 in $[0,T]$, including for $t_1 = 0$, the inequalities

$$-\frac{1}{2}\int_D u^2(x,t)\, dx\Big|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} [L_1(u,u) + L_2(\hat{f},u)]\, dt \le 0. \tag{10}$$

The coefficients a_{ij}, b_i, a_i, a and the free terms f and f_i satisfy conditions (5)–(9). Then

$$|u|_{Q_T} \le C[\|u(x,0)\|_{L_2(D)} + \|\bar{f}\|_{L_2(Q_T)} + \|f\|_{L_{R_4}(Q_T)}], \tag{11}$$

where C is a constant depending on $n, \nu, \mu, \mu_1, R_{ij}, R_3, R_4$.

Theorem 3. For any generalized solution $u(x,t)$ from $\mathring{V}_2^{1,0}$ of problem (1)–(2), the inequality

$$|u|_{Q_T} \le C[\|\psi_0\|_{L_2(D)} + \|\bar{f}\|_{L_2(Q_T)} + \|f\|_{L_{R_4}(Q_T)}]$$

holds, provided that assumptions (5)–(9) are fulfilled with respect to equation (1).

Theorem 4. If conditions (5)–(9) are fulfilled and $\psi_0 \in L_2(D)$, then problem (1), (2) has a solution from $V_2(Q_T)$.

Theorem 5. Under conditions (5)–(9), any generalized solution $u(x,t)$ of problem (1), (2) from $\mathring{V}_2(Q_T)$ belongs to $\mathring{V}_2^{1,1/2}(Q_T)$, and problem (1), (2) is uniquely solvable in $\mathring{V}_2^{1,1/2}(Q_T)$, if $\psi_0(x) \in L_2(D)$.

Theorem 6. If the coefficients of the equation satisfy conditions (5)–(9), then the boundary-value problem for (1), (2) cannot have two distinct generalized solutions from $V_2(Q_T)$.

Theorem 7. Suppose $u(x,t)$ is a generalized solution from $\mathring{V}_2(Q_T)$ of problem (1), (2), and suppose a_{ij}, a_i, and f_i satisfy conditions (5)–(9), while $b_i(x,t), a(x,t)$, and $f(x,t)$ satisfy the conditions

$$\left\|\sum b_i^2,\ a\right\|_{L^*_{(r_1,r_2,l_1)}(Q_T)} \le \mu_1, \qquad l_1 \ge r_2 \ge r_1; \qquad \|f\|_{L^*_{(r_{41},r_{42},l')}(Q_T)} \le \mu_1;$$

$$L_{(r_{11},r_2,l_1)}(Q_T) \in X_{(2-2/l_1)}; \qquad L_{(r_{41},r_{42},l')}(Q_T) \in X_{((n+2)/2-1/l_1)},$$

where

$$L^*_{(m_1,m_2,\rho)}(Q_T) = L_{(\rho,m_2,m_1)}(0 \le t \le T, \widetilde{D}_2, \widetilde{D}_1).$$

Then the function $u(x,t)$ is an element of $\mathring{W}_2^{1,-1/2}(Q_T)$.

Theorem 8. Suppose that for all operators

$$\mathcal{L}^m u = u_t - \frac{\partial}{\partial x_i}(a_{ij}^m u_{x_j} + a_i^m u) + b_i^m u_{x_i} + a^m u, \qquad m = 1, 2, \dots,$$

the conditions of Theorem 4 are satisfied with the same constants. Suppose that $a_{ij}^m(x,t)$, remaining uniformly bounded, converge almost everywhere to a_{ij}, and that the functions $a_i^m, b_i^m, a^m, f_i^m, \psi_0^m$ converge to a_i, b_i, a, f, ψ_0 in the norms of the spaces to which they belong under the conditions of Theorem 4. Then the generalized solutions u^m from $V_2^{1,0}(Q_T)$ of the problems

$$\mathcal{L}^m u \equiv \partial f_i^m/\partial x_i - f^m, \qquad u\big|_{S_T} = 0, \qquad u\big|_{t=0} = \psi_0^m$$

converge strongly in $V_2^{1,0}(Q_T)$ to the generalized solution $u(x,t)$ of the limiting problem (1)–(2).

Theorem 9. Suppose that the coefficients of equation (1) satisfy the conditions

$$\nu\xi_i\xi_i \le a_{ij}\xi_i\xi_j \le \mu\xi_i\xi_i, \qquad \nu,\mu = \text{const} > 0, \quad a_{ij} = a_{ji},$$

$$\|a, f, a_i^2, f_i^2\|_{L(r_1,r_2,l)(Q_T)} < \mu_1, \qquad \|b_i^2\|_{L(\bar r_1,\bar r_2,\bar l)(Q_T)} \le \mu_1,$$

where $(r_1, r_2) \in \Omega^{(\varepsilon)}_{(2-2/l_1)}$, $(\bar r_1, \bar r_2) \in \Omega_{(2-2/\bar l_1)}$.

Then, for every generalized solution $u(x,t)$ from $V_2^{1,0}(Q_T)$ of equation (1), not exceeding k_0 on Γ_T, $\operatorname{vrai\,max} u(x,t)$ is finite and is estimated from above by a constant determined only by $n, k_0, \nu, \mu, \mu_1, r_1, r_2, l_1, \bar r_1, \bar r_2, \bar l_1$.

Theorem 10. Suppose that $u(x,t)$ is a generalized solution from $V_2^{1,0}(Q_T)$ of equation (1), whose coefficients a_{ij}, b_i, a satisfy the conditions

$$\nu\xi_i\xi_i \le a_{ij}\xi_i\xi_j \le \mu\xi_i\xi_i, \qquad \nu,\mu = \text{const} > 0,$$

$$\|b_i^2, a\|_{L(r_1,r_2,l)(Q_T)} \le \mu_1, \qquad (r_1, r_1) \in \Omega_{(2-2/l_1)},$$

$$a(x,t) \ge 0, \qquad a_i = f_i = f = 0.$$

Then for almost all (x,t) in Q_T

$$\min\left\{0, \operatorname*{vrai\,min}_{\Gamma_T} u(x,t)\right\} \le u(x,t) \le \max\left\{0, \operatorname*{vrai\,max}_{\Gamma_T} u(x,t)\right\}.$$

Theorem 11. Suppose that $u(x,t)$ is a generalized solution from $V_2^{1,0}(Q_T)$ of equation (1), whose coefficients and free terms satisfy the conditions of Theorem 9. Then, for any cylinder Q' lying at a positive distance d from Γ_T, the quantity $\operatorname{vrai\,max}_{Q'} |u|$ is estimated from above by a constant depending only on $n, \nu, \mu, \mu_1, r_1, r_2, l_1, \bar r_1, \bar r_2, \bar l_1, d, \|u\|_{L_2(Q_T)}$.

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REFERENCES

1. O. A. Ladyzhenskaya, V. A. Solonnikov, N. U. Ural' tseva, *Linear and quasilinear equations of parabolic type*, "Nauka," 1967.

2. O. A. Ladyzhenskaya, Tr. IV All-Union Mathematical Congress, 1, 1963.

3. A. V. Ivanov, O. A. Ladyzhenskaya et al., Tr. Mat. Inst. im. V. A. Steklova, Academy of Sciences of the USSR, 92 (1965).

4. O. A. Oleinik, Izv. AN SSSR, Ser. Mat., 25, 3 (1961).

5. A. M. Il' in, A. O. Kalashnikov, O. A. Oleinik, UMN, 17, no. 3, 3 (1962).

6. V. P. Il' in, Tr. Mat. Inst. im. V. A. Steklova, Academy of Sciences of the USSR, 66 (1962).

Note: Figure translations are in progress. See original paper for figures.