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# ON DYADIC BICOMPACTS

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**Abstract**

**Full Text**

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**MATHEMATICS**

## A. V. ARKHANGEL'SKII, V. I. PONOMAREV ON DYADIC BICOMPACTS

*(Presented by Academician P. S. Aleksandrov on 15 II 1968)*

In this paper a new approach is given to the proof of the well-known theorem of A. S. Esenin-Vol'pin<sup>(1-3)</sup> on the metrizable of a dyadic bicomact with the first axiom of countability, and new results are obtained. By  $\omega(X)$  we denote the weight of the space  $X$ , by  $|X|$  its cardinality, by  $s(X)$  its density, i.e. the least of the cardinalities of everywhere dense subsets in  $X$ ;  $\chi(x, X)$  is the character of the space  $X$  at the point  $x \in X$ .

**Proposition 1.** *Let  $X$  be a continuous image of  $D^\tau$ , and let  $A \subset X$  be arbitrary. Then there exists a  $Z \subset X$  such that: a)  $Z$  is a continuous image of  $D^t$ ; b)  $A \subseteq Z$ ; c)  $s(A) \geq s(Z)$ .*

**Proof.** Let  $f : D^\tau \rightarrow X$  be a continuous mapping; let  $A^* \subset A$  be a subset everywhere dense in  $A$ , for which  $|A^*| = s(A)$ , and let  $L \subset D^\tau$  be a subspace mapped by  $f$  one-to-one onto  $A^*$  (obviously such a subspace exists). Recall that  $D^\tau$  is a topological group, and take its smallest closed subgroup  $G_L$  containing  $L$ . It is clear that  $s(G_L) \leq |L| = |A^*| = s(A)$ . Consequently,  $s(f(G_L)) \leq s(A)$ . It is well known that the space  $G_L$  is homeomorphic to  $D^{\tau'}$  for some  $\tau' \leq \tau$  (see the direct proof in E. Hewitt's paper<sup>(6)</sup>). Hence one may take  $f(G_L)$  for  $Z$ .

**Remark.** Proposition 1 remains valid if  $X$  is assumed to be the image of some topological group and, instead of a), it is required that  $Z$  be the image of some topological group.

**Lemma 1.** *Let  $X$  be a topological space;  $P$  a class of its subspaces,  $P \ni X$ , and for every  $A \subset X$  there exists an  $\tilde{A} \in P$ ,  $\tilde{A} \supset A$ , whose density  $s(\tilde{A})$  does not exceed the density  $s(A)$  of the space  $A$ . If  $s(X) > \tau_1$ , then there exists  $Y_1 \in P$  for which  $s(Y_1)$  is a regular cardinal number greater than  $\tau_1$ .*

**Proof.** Put

$$P(\tau_1) = \{Y \in P : \text{if } Y' \subseteq X \text{ and } Y' \supseteq Y, \text{ then } s(Y') \geq s(Y) > \tau_1\},$$

and let  $Y_1 \in P(\tau_1)$  be chosen so that  $s(Y_1) \leq s(Y)$  for every  $Y \in P(\tau_1)$ . Then  $s(Y_1)$  is a regular cardinal number.

Suppose the contrary. Choose  $A_1$  everywhere dense in  $Y_1$  so that  $|A_1| = s(Y_1)$ , and let

$$A_1 = \bigcup \{A_1^\alpha : \alpha \in M\},$$

where  $|M| < |A_1|$  and  $|A_1^\alpha| < |A_1|$ . For every  $A_1^\alpha$ ,  $\alpha \in M$ , find  $\tilde{A}_1^\alpha \in P$  of least density satisfying the condition of the lemma:

$$s(\tilde{A}_1^\alpha) \leq |A_1^\alpha|, \quad \tilde{A}_1^\alpha \supset A_1^\alpha.$$

Then from  $s(\tilde{A}_1^\alpha) < |A_1| = s(Y_1)$  and  $\tilde{A}_1^\alpha \in P$  it follows that  $\tilde{A}_1^\alpha \notin P(\tau_1)$ , whence  $s(\tilde{A}_1^\alpha) \leq \tau_1$  by the choice of  $\tilde{A}_1^\alpha$ . Since the last relation holds for all  $\alpha \in M$ , we have

$$s\left(\left[\bigcup \{\tilde{A}_1^\alpha : \alpha \in M\}\right]\right) \leq \max(\tau_1 |M|) < s(Y_1).$$

But

$$\left[\bigcup \{\tilde{A}_1^\alpha; \alpha \in M\}\right] \supseteq Y_1$$

and  $Y_1 \in P(\tau_1)$ —a contradiction.\*  $P(\tau_1) \neq \Lambda$ , for  $X \in P(\tau_1)$ .

We shall say that a topological space **satisfies Shanin's condition** <sup>(5)</sup> if every family, completely ordered by inclusion (decreasing), of its nonempty open sets whose intersection is empty contains a countable cofinal part.

**Lemma 2.** *If a topological space  $X$  satisfies Shanin's condition and its density  $s(X)$  is regular and is not equal to  $\aleph_0$ , then*

$$s(X) \leq \partial(x, X)$$

\*for some point  $x \in X^{**}$ .\*

\* From Proposition 1 and Lemma 1 it follows that the density of every (dyadic) bicomact  $X$  is the least upper bound of those regular cardinal numbers which are densities of (dyadic) bicomacts contained in  $X$ . An analogous assertion is true for arbitrary spaces.

\*\* By  $\partial(x, X)$  is denoted the least cardinal number  $\tau$  such that, if  $x \in [M]$ , where  $M \subset X$ , then there exists  $M' \subset M$  with the properties  $|M'| \leq \tau$  and  $x \in [M']$ .

**Proof.** Let  $A \subset X$ , let  $A$  be everywhere dense in  $X$ , and let  $|A| = s(X) = \tau$ . We shall assume that the set  $A$  is well ordered according to the minimal ordinal type of the cardinal number  $\tau$  (by the relation  $<$ ). For each  $y \in A$  put

$$A(y) = \{y' \in A : y' < y\}$$

and

$$B(y) = [A(y)].$$

Then, if  $A_0 \subset A$  and  $|A_0| < \tau$ , there exists  $y_0 \in A$  for which  $A(y_0) \supset A_0$  (by the regularity of the cardinal  $\tau$ ). Suppose now that  $\partial(x_0, X) < s(X) = \tau$  for

some  $x_0 \in X$ . From what was said above (and from the definition of  $\partial(x_0, X)$ ) it follows that there exists  $y'_0 \in A$  for which  $[A(y'_0)] \ni x_0$ , i.e.  $B(y'_0) \ni x_0$ . If we assume that the relation  $\partial(x, X) < \tau$  holds for all  $x \in X$ , then we obtain

$$\bigcup \{B(y) : y \in A\} = X.$$

But from  $|A(y)| < |A| = s(X)$  (true for all  $y \in A$ ) it follows that  $B(y) \neq X$ . Hence  $\{G(y) : y \in A\}$ , where  $G(y) = X \setminus B(y)$ , is a decreasing (since  $B(y) \subseteq B(y')$  for  $y < y'$ ) sequence of nonempty open subsets of  $X$ , not cofinal with the countable one, with empty intersection.

**Definition 1.** By  $t(X)$ , where  $X$  is a topological space, we shall denote the least cardinal number  $\tau$  such that, whatever the set  $A \subset X$  and the point  $x \in [A]$ , there exists  $A' \subset A$  for which  $|A'| \leq \tau$  and  $x \in [A']$ . We shall call  $t(X)$  the **tightness** of the space  $X$ . It is obvious that, if  $Y \subset X$ , then  $t(Y) \leq t(X)$ .

**Theorem 1.** Let  $X$  be a space satisfying the following condition:

(III). If  $A \subset X$ , then there exists a  $Y \subset X$ ,  $Y \supset A$ , satisfying Shanin's condition, for which  $s(A) \geq s(Y)$ .

Then

$$t(X) \geq s(X).$$

**Proof.** By contradiction: suppose  $s(X) > t(X)$ . Denote by  $P$  the collection of all subspaces of the space  $X$  satisfying Shanin's condition, and put  $\tau_1 = t(X)$ . On the basis of Lemma 1 we conclude that there exists  $Y_1 \subset X$ ,  $Y_1 \in P$ , for which  $s(Y_1) > \tau_1 = t(X)$ , and  $s(Y_1)$  is a regular cardinal number. Then

$$s(Y_1) > t(Y_1),$$

which contradicts Lemma 2. The theorem is proved.\*

**Definition 2.** A space  $X$  is called a  $\pi$ -space if two conditions are satisfied for it: a) for each  $A \subset X$  there exists a subspace  $\bar{A} \subset X$  containing it, satisfying Shanin's condition and having density not exceeding the density of  $A$ ; b) the  $\pi$ -weight of the space  $X$  is equal to the weight of  $X$ .

As is known, all dyadic bicomacts satisfy conditions a) and b), and consequently all of them are  $\pi$ -spaces.

It is known that the  $\pi$ -weight\*\* of a space  $X$  (unlike weight, generally speaking) is very simply connected with the density of  $X$  and the upper bound of the characters of points of  $X$ —namely, the  $\pi$ -weight of  $X$  does not exceed the maximum of the following two numbers.

**Theorem 2.** The weight of a  $\pi$ -space  $X$  is equal to

$$\tau = \sup\{\chi(x, X) : x \in X\}.$$

**Proof.** We have

$$\sup\{\chi(x, X) : x \in X\} \geq t(X).$$

By Theorem 1,

$$t(X) \geq s(X).$$

Thus,  $\tau \geq s(X)$ . But

$$\omega(X) = \pi(X) \leq \max(\tau, s(X)).$$

Consequently,  $\omega(X) \leq \tau$ . Since always

$$\omega(X) \geq \sup\{\chi(x, X) : x \in X\},$$

we conclude that

$$\omega(X) = \tau.$$

**Corollary 1** (A. S. Esenin-Vol' pin). *The weight of a dyadic bicomact does not exceed the upper bound of the characters of its points.*

\* We note that we have proved the general assertion: if a cardinal-valued function  $\delta$  is defined on the class of all bicomacts, is monotone with respect to closed subspaces, and  $\delta(X) \geq s(X)$  for any (dyadic) bicomact  $X$  whose density  $s(X)$  is regular, then  $\delta(X) \geq s(X)$  for any (dyadic) bicomact  $X$  (such is the tightness  $t(X)$ ).

\*\* The  $\pi$ -weight of a space  $X$  is the least cardinal number equal to the cardinality of a system of open subsets of  $X$  that is dense in  $X$ . A system  $\sigma$  of open subsets of  $X$  is dense in  $X$  if for any open set  $\Gamma \subseteq X$  there exists  $U \in \sigma$  for which  $U \subseteq \Gamma$  (see (7)).

**Corollary 2.** If a topological space  $X$  and each of its closed subspaces satisfy Shanin' s condition, then

$$s(X) \leq \sup\{\partial(x, X) : x \in X\}.$$

**Lemma 3.** If a space  $X$  is separable, then it also satisfies Shanin' s condition.

From Lemma 3 and Corollary 2 there further follows

**Theorem 3.** In order that a Fréchet-Urysohn space and each of its closed subspaces satisfy Shanin' s condition, it is necessary and sufficient that it be hereditarily separable with respect to closed subsets.

**Theorem 4.** In order that a space  $X$  and each of its subspaces satisfy Shanin' s condition, it is necessary and sufficient that  $X$  be hereditarily separable with respect to all subspaces.

**Proof.** a) Let  $X$  and each of its subsets satisfy Shanin' s condition. Suppose that there is a nonseparable subspace in  $X$ . Then, by Lemma 1, there exists a subspace  $\bar{A} \subset X$  whose density  $s(\bar{A})$  is equal to an uncountable regular cardinal number; but then there exists a set  $B \subseteq \bar{A}$  that does not satisfy Shanin' s condition. The contradiction obtained proves that  $X$  is hereditarily separable.

b) The converse assertion follows immediately from Lemma 3.

**Remark 1.** It is not hard to verify that a space  $X$  satisfies Shanin's condition if and only if every system  $\sigma$  of regular cardinality, consisting of sets open in  $X$ , contains a subsystem of the same cardinality with nonempty intersection.

**Remark 2.** It is easy to see that Suslin's condition is formally weaker than Shanin's condition. Therefore it is natural to pose the following problem:

**Is every bicomcompact space with the first axiom of countability, satisfying Suslin's condition hereditarily with respect to closed subsets, separable?**

This question, as we shall now show, is connected in one direction with the well-known Suslin problem. Indeed, a nonseparable Suslin continuum, if it exists, satisfies Suslin's condition hereditarily with respect to closed subsets, satisfies the first axiom of countability, and its weight and  $\pi$ -weight are equal\*. The Suslin continuum itself even satisfies Shanin's condition. However, we do not know whether or not our problem is equivalent to the Suslin problem.

**Definition 3.** A point  $x \in X$  will be called a **point of extremal accumulation** for a set  $M \subset X$  if

$$\inf\{Ox \cap M : Ox \text{ is a neighborhood of } x\} > \inf\{Oy \cap M : Oy \text{ is a neighborhood of } y\}$$

for every  $y \neq x, y \in X$ .

**Definition 4.** We shall say that  $X$  is a  $\chi$ -space if from  $x \in X, M \subset X, x \in [M]$  it follows that there exists  $M' \subset M$  for which  $x$  is the unique point of extremal accumulation in  $X$ .

An example of a non- $\chi$ -space is  $\beta N$ . Every Fréchet-Urysohn space is a  $\chi$ -space\*\*. It is clear that every closed subspace of a  $\chi$ -space is itself a  $\chi$ -space. Therefore not every dyadic bicomcompactum is a  $\chi$ -space (some of them contain  $\beta N$ ).

Of fundamental importance is the following obvious assertion.

**Proposition 2.** If  $X$  is a  $\chi$ -space, then  $|X| \leq 2^{s(X)}$ .

In proving the following assertion we assume that

$$2^{\tau_1} > 2^{\tau_2},$$

if  $\tau_1 > \tau_2$  (which is weaker than the generalized continuum hypothesis).

\* Cf. these conditions with the references of the direct Theorem 3.

\*\* The space of all transfinite ordinals not exceeding the first uncountable is a  $\chi$ -space, but not a Fréchet-Urysohn space.

**Theorem 5.** If a bicomcompactum  $X$  is a  $\chi$ -space and for every  $A \subset X$  there exists  $\tilde{A} \supset A, s(\tilde{A}) \geq |A|$ , satisfying Shanin's condition, then

$$[\{x \in X : \chi(x, X) \leq t(X)\}] = X.$$

**Proof.** It has already been proved that  $t(X) \geq s(X)$ . Moreover,  $\omega(X) \geq t(X)$ .  
If

$$X \neq [\{x \in X : \chi(x, X) \leq t(X)\}],$$

then, by the Čech-Pospíšil theorem <sup>(8,9)</sup>,

$$|X| \geq 2^{t(X)+1}.$$

But by Proposition 2,

$$|X| \leq 2^{s(X)},$$

i.e.

$$2^{t(X)+1} \leq 2^{s(X)}.$$

But  $t(X) + 1 > s(X)$ —a contradiction.

**Remark.** The advantage of the formulations and arguments given above is that the topological background of the equality  $t(X) = \omega(X)$  is touched upon rather deeply. However, if one restricts oneself to dyadic bicompacta, the final result can be obtained without using the continuum hypothesis at all.

**Definition 5.** The **weak tightness**  $t_0(X)$  of a topological space  $X$  is the least of the cardinal numbers  $\tau$  for which:

(\*) if  $M \subset X$  and  $M \neq [M]$ , then there exist  $M' \subset M$  and  $x' \in X$  such that

$$|M'| \leq \tau \quad \text{and} \quad x' \in [M'] \setminus M.$$

It is clear that always

$$t_0(X) \leq t(X) \leq \sup\{\chi(x, X) : x \in X\} \leq \omega(X).$$

Weak tightness is sometimes more convenient than tightness, for the following is true.

**Proposition 3.** *If  $Y$  is a quotient space of a space  $X$ , then*

$$t_0(Y) \leq t_0(X).$$

Now the first of the following two theorems follows from the second.

**Theorem 6.** *If a dyadic bicompactum  $Y$  is a quotient space of a space  $X$ , then*

$$\omega(Y) \leq \sup\{\chi(x, X) : x \in X\}.$$

**Theorem 7.** *For every dyadic bicompactum  $X$ ,*

$$t_0(X) = t(X) = \sup\{\chi(x, X) : x \in X\} = \omega(X).$$

**Corollary.** *If for a dyadic bicompactum  $X$*

$$t_0(X) = \aleph_0,$$

then also

$$\omega(X) \leq \aleph_0.$$

This contains, in particular, the result of B. A. Efimov: a dyadic bicomactum which is a Fréchet-Urysohn space is metrizable.

**Proposition 4\*.**  $t_0(X) \leq \tau$  if and only if  $X$  is a quotient space of a space  $\tilde{X}$  of local cardinality  $\leq \tau$  (briefly:  $|\tilde{X}|_{\text{loc}} \leq \tau$ ).

This characterization is useful: since the property of having local cardinality  $\leq \tau$  is inherited by all subspaces, and a quotient mapping on the full preimage of an open set induces a quotient mapping, we derive from Proposition 4:

**Proposition 5.** If  $U$  is open in  $X$ , then

$$t_0(U) \leq t_0(X).$$

Theorem 6 can be given the following form:

**Theorem 6'.** If a dyadic bicomactum  $Y$  is a quotient space of a space  $X$  whose local cardinality does not exceed  $\tau$ , then also

$$\omega(Y) \leq \tau.$$

**Question 1.** Does there exist a bicomactum  $X$  which is simultaneously a  $\chi$ -space and a  $\pi$ -space, for which

$$t(X) \neq \omega(X)?$$

**Question 2.** Does there exist a nonmetrizable dyadic bicomactum which is a  $\chi$ -space?

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- \* This proposition underlies the proof of Proposition 3.

*Note: Figure translations are in progress. See original paper for figures.*

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