THREE THEOREMS ON NONLINEAR EQUATIONS

MATHEMATICS

1968

SovietRxiv

Abstract

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UDC 517.947.35

MATHEMATICS

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THREE THEOREMS ON NONLINEAR EQUATIONS

WITH MONOTONE OPERATORS

(Presented by Academician V. I. Smirnov on 11 III 1968)

The paper contains three theorems. The first theorem establishes the convergence of the method of least squares for equations with monotone operators and represents a further development of the results of S. G. Mikhlin and A. Langenbach.

The second theorem gives conditions for the solvability of equations with monotone operators (cf. ([5–8])). In contrast to existing works, reflexivity of the space is not assumed in it. Moreover, monotonicity is understood in a more general sense. The third theorem gives sufficient conditions for uniform convexity of a space in terms of monotone operators.

1. Let E be a real reflexive separable Banach space; let E^* be its conjugate; suppose that linear combinations of the elements $x_1, \dots, \dots, x_n, \dots$ form an everywhere dense set $N \subseteq E$, and that the elements x_i are linearly independent in any finite number. Let $N \subseteq M \subseteq E$, and let the operator F act from the set M into E^*. As is known ([1]), the method of least squares for solving the equation $F(x) = \theta$ consists in seeking an absolute minimum of the function

$$\left\| F\left(\sum_{i=1}^{n} a_i x_i\right)\right\|$$

of n real variables $a_1, \dots, a_n$, for fixed $x_1, \dots, x_n$. If the minimum is attained at the element

$$u_n^0 = \sum_{i=1}^{n} a_i^0 x_i$$

(not necessarily unique), then u_n^0 is an approximate solution obtained by the method of least squares. The sequence $\{x_i\}$ is called F-complete ([1]) in E if, for every $\varepsilon > 0$, there exist a natural number n and real numbers $\alpha_1, \dots, \alpha_n$ such that

$$\left\| F\left(\sum_{i=1}^{n} \alpha_i x_i\right) \right\| < \varepsilon.$$

Theorem 1. Let the operator F satisfy the condition of strong monotonicity

$$(x - y, F(x) - F(y)) \geq \gamma(\|x - y\|)\|x - y\|, \quad \forall x,\ y \in M,$$

where $\gamma(t)$ is a continuous strictly increasing function for $t \geq 0$, $\gamma(0) = 0$,

$$\lim_{t \to +\infty} \gamma(t) = +\infty.$$

Suppose that at least one of the conditions I, II holds:

I. The restriction of the operator F to every finite-dimensional subspace of M is a continuous operator; there exists $x^* \in M$, $F(x^*) = \theta$; the sequence $\{x_i\}$ is F-complete.

II. $M = E$ and F is a continuous operator (one can prove that in this case there exists a unique solution x^* of the equation $F(x) = \theta$).

Then the approximate solutions u_n^0, obtained by the method of least squares, exist and

$$\lim_{n \to \infty} \|u_n^0 - x^*\| = 0.$$

Proof. If condition I is fulfilled, then by virtue of F-completeness there exists a sequence of elements

$$u_n = \sum_{i=1}^{n} a_i x_i$$

such that $F(u_n) \to \theta$.

If condition II is fulfilled, then, since N is everywhere dense in E, there also su-

there exists a sequence of elements $u_n = \sum_{i=1}^{n} a_i x_i$ such that $u_n \to x^*$, and therefore $F(u_n) \to F(x^*) = \theta$. Thus, in any case $F(u_n) \to \theta$. Since $\|F(u_n^0)\| \leq \|F(u_n)\|$, it follows that $F(u_n^0) \to \theta$. From the condition of strong monotonicity it immediately follows that $\lim_{n \to \infty} \|u_n^0 - x^*\| = 0$.

2. Let X be a Banach space, and let Z be a real Banach space with cone K, i.e. K is a closed convex set such that, for $x \in K$, $\forall \lambda \geq 0$ we have $\lambda x \in K$; for $x \neq \theta_Z$ we have $(-x) \notin K$ (see [4], p. 13). Suppose that in X

there is defined an operation of multiplication $z = xy$, $\forall x, y \in X$, linear in each variable, with values of the product in Z. Let multiplication and the norms possess the properties $\|xy\|_Z \le \|x\|_X \|y\|_X$, $\|x^2\|_Z = \|x\|_X^2$, $x^2 \ge \theta_Z$, $\forall x, y \in X$, and let the norm in Z be monotone, i.e. if $w \ge z \ge \theta_Z$, then $\|w\|_Z \ge \|z\|_Z$, $\forall z, w \in Z$. A pair of spaces X, Z satisfying the requirements listed above will be called an admissible pair. For example, if X is Hilbert, $Z = R^1$, $K = \{z : z \ge 0,\ z \in R^1\}$, and xy is the scalar product, then X, Z is an admissible pair of spaces. Two spaces of continuous functions $X \equiv Z \equiv C[a, b]$ with the cone of nonnegative functions and the usual multiplication give another example of an admissible pair of spaces.

Theorem 2. Let X, Z be an admissible pair of spaces, and let the operator $F(x)$, acting from X into X, in each ball $D_r\{x : \|x\| \le r,\ x \in X\}$ satisfy the Lipschitz and strong monotonicity conditions:

$$[F(x) - F(y)]^2 \le m^2(r)(x - y)^2, \qquad \forall x, y \in D_r; \tag{1}$$

$$[F(x) - F(y)](x - y) - k(r)(x - y)^2 \ge \theta_Z, \qquad \forall x, y \in D_r; \tag{2}$$

$$(x - y)[F(x) - F(y)] - k(r)(x - y)^2 \ge \theta_Z, \qquad \forall x, y \in D_r, \tag{3}$$

where $k(r) > 0$, $m(r) > 0$, $\forall r > 0$, and $\lim_{r\to+\infty} rk(r) = +\infty$.
Then the operator $F(x)$ realizes a homeomorphic mapping of the space X onto X.

Proof. From inequality (1) it follows that $F(x)$ satisfies the usual Lipschitz condition $\|F(x) - F(y)\|_X \le m(r)\|x - y\|_X$, $\forall x, y \in D_r$, and from inequality (2) we have $k(r)\|x - y\|_X^2 \le \|F(x) - F(y)\|_X \|x - y\|_X$, $\forall x, y \in D_r$.

Therefore $k(r) \le m(r)$, $\forall r > 0$. Since $\lim_{r\to+\infty} rk(r) = +\infty$, there is $r_0 > 0$ such that the inequality holds:

$$\|F(\theta)\|_X \left[1 + \sqrt{1 - k^2(r_0)/m^2(r_0)}\right] < r_0 k(r_0). \tag{4}$$

Put

$$\Phi(x) = x - \frac{k(r_0)}{m^2(r_0)} F(x) \tag{5}$$

and consider the sequence

$$x_n = \Phi(x_{n-1}), \qquad x_0 = \theta. \tag{6}$$

By a simple calculation (using inequalities (1), (2), (3)) we obtain

$$[\Phi(x) - \Phi(y)]^2 \leq (x-y)^2[1 - k^2(r_0)/m^2(r_0)], \qquad \forall x, y \in D_{r_0}.$$

Putting $\mu_0(r_0) = \sqrt{1 - k^2(r_0)/m^2(r_0)}$, from the last inequality we obtain

$$\|\Phi(x) - \Phi(y)\|_X \leq \mu_0(r_0)\|x - y\|_X, \qquad \forall x, y \in D_{r_0}, \tag{7}$$

where $\mu_0(r_0) < 1$.

From inequalities (4), (5), (6) we have:

$$\|x_1\|_X = \|\Phi(\theta)\|_X \leq \frac{k(r_0)}{m^2(r_0)} \frac{r_0 k(r_0)}{1 + \sqrt{1 - k^2(r_0)/m^2(r_0)}} = [1 - \mu_0(r_0)]r_0, \tag{8}$$

i.e. $x_1 \in D_{r_0}$. Assuming that $x_1, \dots, x_{n-1} \in D_{r_0}$, we find

$$\|x_n\|_X \leq \|x_n - x_{n-1}\|_X + \|x_{n-1} - x_{n-2}\|_X + \dots + \|x_1 - x_0\|_X \leq$$

$$\leq \sum_{i=1}^{n} \mu_0^{i-1}(r_0)\|x_1 - x_0\|_X.$$

Hence, and from the estimate for x_1, we have

$$\|x_n\|_X \leq \sum_{i=1}^{n} \mu_0^{i-1}(r_0)[1 - \mu_0(r_0)]r_0 = [1 - \mu_0^n(r_0)]r_0 < r_0,$$

so that, by induction, all $x_n \in D_{r_0}$. Using inequality (7), in the standard way one can prove that x_n is a fundamental sequence, and therefore there exists $\lim_{n\to\infty} x_n = x^*$. Taking into account the continuity of $\Phi(x)$, we obtain from this:

$$x^* = \Phi(x^*) = x^* - \frac{k(r_0)}{m^2(r_0)} F(x^*),$$

or $F(x^*) = \theta$. If φ is an arbitrary element of X, then the operator $F_1(x) = F(x) - \varphi$, evidently, satisfies all the conditions of the theorem, and therefore the equation $F_1(x) = \theta$ is solvable in X. Thus, the range of the operator $F(x)$ is the whole space X. The fact that $F(x)$ realizes a homeomorphic mapping is obtained quite simply if one uses inequality (2). The theorem is proved.

3. Let E be a real Banach space; E^*, its conjugate; and let the operator $F : E \to E^*$ be strongly monotone in the following sense:

$$(x - y, F(x) - F(y)) \geq \gamma(\|x - y\|)\|x - y\|, \qquad \forall x, y \in E. \tag{9}$$

(The notation (x, z) means the value of the linear functional $z \in E^*$ at the element $x \in E$.) Such operators have been studied in a number of works. The question naturally arises whether such an operator exists in every space. A partial answer to the question is given by

Theorem 3. *Suppose there exists an operator F defined on the whole space E and satisfying inequality (9). Suppose, moreover, that the following conditions are fulfilled: a) F is the gradient of a functional $f : E \to R^1$; b) the functional $f(x)$ is uniformly continuous on every bounded set of the space E; c) for any fixed $x, h \in E$ the function $(F(x + th), h)$ is continuous in t; d) the real function γ of a nonnegative argument is such that the function*

$$\nu(R) = \int_0^1 \gamma(tR)\, dt$$

is continuous, strictly increasing for $R \geq 0$, $\nu(0) = 0$, and $\lim_{R\to\infty} \nu(R) = +\infty$.

Then in the space E one can introduce a norm equivalent to the original one, so that in the new norm the space E will be uniformly convex.

Proof. Carrying out simple transformations, and using inequality (9) (cf. (2)), we obtain

$$\frac{1}{2}f(x) + \frac{1}{2}f(y) - f\left(\frac{1}{2}(x + y)\right) \geq \frac{1}{4}\nu(\|x - y\|)\|x - y\|, \qquad \forall x, y \in E. \tag{10}$$

Without loss of generality, one may assume that the functional f is even and $f(\theta) = 0$; otherwise one may introduce a new functional $\varphi(x) = \frac{1}{2}[f(x) + f(-x)] - f(\theta)$, which also satisfies inequality (10). Setting $y = -x$ in (10), we find $f(x) \geq \frac{1}{2}\nu(2\|x\|)\|x\|$, whence it follows that

$$\lim_{\|x\|\to\infty} f(x) = +\infty. \tag{11}$$

It is not difficult to see that the functional f is nondecreasing along every ray issuing from zero. Consider now the set

$$K = \{x : f(x) \leq 1,\ x \in E\}. \tag{12}$$

By virtue of inequality (10), K is a convex set. Since the functional f is continuous and $f(\theta) = 0$, θ is an interior point of the set K. Consider the Minkowski functional $m(x)$ of the set K: $m(x) = \inf a_x,\ f(x/a_x) \leq$

$\leqslant 1,\ a_x > 0$. It is known that $m(x) \geqslant 0,\ m(x) < \infty,\ m(\lambda x) = \lambda m(x),\ m(x + y) \leqslant m(x) + m(y),\ \forall x, y \in E,\ \forall \lambda \geqslant 0$. By virtue of the evenness of the functional f we have $m(\lambda x) = |\lambda| m(x)$ for all real λ. From the above-mentioned properties of the functional f it follows easily that $m(x) = 0$ if and only if $x = \theta$. Thus, $m(x)$ is a new norm in E. It is easy to verify that it is equivalent to the original norm $\|\cdot\|$. We shall now show that, in the norm $m(x)$, the space E is uniformly convex. Let

$$m(x_n) \leqslant 1, \quad m(y_n) \leqslant 1, \quad m(1/2(x_n + y_n)) \to 1. \tag{13}$$

From these relations it follows that

$$m(x_n) \to 1, \quad m(y_n) \to 1. \tag{14}$$

We shall now show that if $m(x_n) \to 1$, then also $f(x_n) \to 1$. Let $m(x_n) \to 1$. The inequality $f(x_n) > 1$ is impossible, for $1/m(x_n) \geqslant 1$, and therefore, by virtue of the nondecrease of the functional $f(x)$ along each ray issuing from θ, we have $f(x_n/m(x_n)) > 1$, which contradicts the definition of $m(x)$. Thus, $f(x_n) \leqslant 1$. Suppose that, for some subsequence $\{x_{n_k}\}$, the inequality $f(x_{n_k}) \leqslant \beta_0 < 1$, where $\beta_0 > 0$, holds. Since $\lim_{n_k \to \infty} m(x_{n_k}) = 1$, for the given $\delta_1 > 0$ there is an N such that for all $n_k \geqslant N$ we shall have

$$\left|1 - 1/m(x_{n_k})\right| < \delta_1. \tag{15}$$

The set K is bounded by virtue of relation (11). Therefore there exists an $M > 0$ such that $\|x_{n_k}\| \leqslant M$ for all n_k. By virtue of the uniform continuity of f on the set K, for the given $\varepsilon > 0,\ \varepsilon + \beta_0 < 1$, there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in K,\ \|x - y\| < \delta$. Taking $\delta_1 = \delta/M$ in (15), we obtain for all $n_k \geqslant N$

$$\|x_{n_k} - x_{n_k}/m(x_{n_k})\| < M\delta_1 = \delta.$$

Since $x_{n_k},\ x_{n_k}/m(x_{n_k}) \in K$, by the uniform continuity of f on K we have, for all $n_k \geqslant N$,

$$|f(x_{n_k}) - f(x_{n_k}/m(x_{n_k}))| < \varepsilon,$$

whence

$$f(x_{n_k}/m(x_{n_k})) < f(x_{n_k}) + \varepsilon \leqslant \beta_0 + \varepsilon,$$

which contradicts the definition of $m(x_{n_k})$. Thus, $\lim_{n \to \infty} f(x_n) = 1$. Hence, from relations (13), (14), it follows that

$$f(x_n) \to 1, \quad f(y_n) \to 1, \quad f(1/2(x_n + y_n)) \to 1.$$

Putting $x = x_n,\ y = y_n$ in inequality (10), we obtain

$$\lim_{n\to\infty} \nu(\|x_n - y_n\|)\|x_n - y_n\| = 0,$$

whence it follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

In view of the equivalence of the norms $\|x\|$ and $m(x)$, from this we find that

$$\lim_{n\to\infty} m(x_n - y_n) = 0.$$

This proves the uniform convexity of the space E.

Let us note that, in fact, we have proved a stronger assertion: if in the Banach space E there exists a functional $f(x)$ satisfying the inequality

$${}^1\!/_2 f(x) + {}^1\!/_2 f(y) - f({}^1\!/_2(x+y)) \geqslant c(\|x-y\|), \quad \forall x, y \in E,$$

where $c(t)$ is a continuous strictly increasing function of the nonnegative argument t, $c(0) = 0$, and

$$\lim_{t\to+\infty} c(t) = +\infty,$$

and if $f(x)$ is uniformly continuous on every bounded set in E, then there exists a norm in E, equivalent to the original one, in which the space E will be uniformly convex.

In conclusion, the author expresses his gratitude to S. G. Mikhlin for discussing the work and for valuable advice.

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Received
21 II 1968

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Note: Figure translations are in progress. See original paper for figures.