ON BICOMPACTA WITH NONCOINCIDING DIMENSIONS

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In 1935 P. S. Aleksandrov raised the question whether, for bicompacta, the dimension defined by means of coverings (dim) and the small inductive dimension (ind) coincide. A. G. Lund ([2]), and then O. V. Lokutsievskii ([1]), constructed bicompacta that are one-dimensional in the sense of dim and two-dimensional in the sense of ind. At the same time, the bicompactum S of O. V. Lokutsievskii was considerably simpler than Lund' s bicompactum; moreover, the bicompactum S decomposed into the sum of two such bicompacta S_1 and S_2 that $\operatorname{ind} S_1 = \operatorname{ind} S_2 = 1$. Thus, O. V. Lokutsievskii' s example also showed that the sum theorem for the small inductive dimension does not hold in bicompacta. With the aid of a sufficiently general construction, P. Vopenka ([3]) constructed, for arbitrary natural numbers m and n ($1 \leq m < n$), such bicompacta X_m^n that $\dim X_m^n = m$, while $\operatorname{ind} X_m^n = n$. All these bicompacta did not satisfy the first axiom of countability, and the dimensions there differed only on a nowhere dense set (namely, on the set of those points which do not have a countable fundamental system of neighborhoods).

Below we shall construct a separable bicompactum B with the first axiom of countability such that $\dim B = 2$ and $3 \leq \operatorname{ind}_b B \leq 4$ for every point $b \in B$. One sufficiently general method for constructing examples of this kind will be indicated.

§ 1. **Construction of the space B.** Denote by J the half-interval $(0, 1]$ of the number line. Denote by I_n the segment $[1/(n+1), 1/n]$ of the number line. Then

$$J = \bigcup_{n=1}^{\infty} I_n.$$

Let $g : J \to T^2$ be such a continuous mapping of the half-interval J onto the torus T^2 (instead of the torus one may take any two-dimensional connected and locally connected compactum) that each segment I_n is mapped by g onto the whole torus T^2. Such a mapping g is easy to construct, starting, for example, from the standard Peano mapping of a segment onto a square.

Let now S^2 be the two-dimensional sphere of diameter 1 (it may be regarded as a subset of three-dimensional Euclidean space), and let x be some point of the sphere S^2. Define a mapping $h_x : S^2 \setminus \{x\} \to T^2$ by the equality

$$h_x(x') = g(\rho(x, x')).$$

Here $\rho(x, x')$ denotes the distance from the point x to the point x'. The function ρ continuously maps the sphere with the point x removed, $S^2 \setminus \{x\}$, onto the half-interval J, and the mapping h_x is continuous as the superposition of the two continuous mappings ρ and g.

The fundamental property of the mapping h_x. For every nonempty open set $W \subset T^2$ and every neighborhood U of the point x, the set $U \cap h_x^{-1} W$ contains a closed set F which separates the sphere S^2 into such open sets D and E that $x \in D \subset U$.

Let $B = S^2 \times T^2$. We shall denote points of the set B by pairs (x, y), where $x \in S^2$, $y \in T^2$. Define on the set B a topology τ, relative to which B will be the desired space. A base of the topology τ is formed by all sets of the form $O(U, x, W)$, where U is an open subset of the sphere S^2, $x \in U$, W is an open subset of the torus T^2, and

$$O(U, x, W) = (\{x\} \times W) \cup \{(U \cap h_x^{-1} W) \times T^2\} = W^x \cup U(W^x).$$

It is easy to verify that the family $\{O(U, x, W)\}$ forms a base for some topology.

Obviously, the topology τ is different from the product topology of S^2 on T^2. At the same time the topology τ induces on each layer $\{x\} \times T^2$ the ordinary topology of the torus T^2.

§ 2. **Basic properties of the space B.**

Theorem 1. *The space B is a separable bicompactum with the first axiom of countability such that* $\dim B = 2$ *and* $3 \le \operatorname{ind}_b B \le 4$ *for every point* $b \in B$.

Outline of the proof. 1. It is easy to verify that the space B is a separable bicompactum with the first axiom of countability. Moreover, the projection $\pi : B \to S^2$ of the product $B = S^2 \times T^2$ onto the first factor is a continuous and irreducible mapping.

2. Let b be an arbitrary point of the bicompactum B. The inequality $3 \le \operatorname{ind}_b B \le 4$ follows from the following assertion.

Lemma. *Let G be an open subset of the bicompactum B such that $B \setminus [G] \neq \emptyset$. Then there exists a point $x \in S^2$ such that $\{x\} \times T^2 = \pi^{-1}(x) \subset \operatorname{Fr} G$.*

Proof. Denote by K the small image of the set G under the mapping π:

$$K = \pi^{\#} G = \{x \in S^2 \mid \pi^{-1}(x) \subset G\},$$

$$L = \pi^{\#}(B \setminus [G]), \qquad M = S^2 \setminus K \cup L.$$

The sets K and L, by virtue of the irreducibility of the mapping π, are disjoint nonempty open subsets of the sphere S^2. Hence the set M separates S^2, and $\dim M = \operatorname{ind} M \geq 1$. Let x be an arbitrary point of the set M for which $\operatorname{ind}_x M \geq 1$. Using the basic property of the mapping h_x, it is easy to show that $\pi^{-1}(x) \subset \operatorname{Fr} G$.

3. The proof of the inequality $\operatorname{ind}_t B \leq 4$ reduces to the sufficiently cumbersome verification that the small inductive dimension of the boundary of any basic set $O(U, x, W)$ is at most 3. Whether in fact the small inductive dimension of the bicompactum B is equal to 3 or to 4 has not been established.

4. $\dim B = 2$. Since the space B contains, as a closed subset, a space homeomorphic to the two-dimensional torus, $\dim B \geq 2$. Let ω be an arbitrary cover of the bicompactum B. Since the mapping $\pi : B \to S^2$ is closed, there exists a finite set of points $x_1, \dots, x_l$ of the sphere S^2 such that the preimage $\pi^{-1}(x)$ of every point $x \neq x_j$, $j = 1, \dots, l$, is contained in some element of the cover ω together with the indicated neighborhood $\pi^{-1}(U_x)$. For each layer $\{x_j\} \times T^2 = \pi^{-1}(x_j)$, $j = 1, \dots, l$, one can choose a system β_j, consisting of basic sets of the space B and inscribed in the cover ω, such that the multiplicity of the system $\beta_j \leq 3$ and the body of the system β_j coincides with the set $U_j \times T^2 = \pi^{-1} U_j$, where U_j is a neighborhood of the point x_j. This can be done so that the sets U_j are pairwise disjoint and the multiplicity of the system

$$\beta = \bigcup_{j=1}^{l} \beta_j$$

in some neighborhood G of the set $B \setminus \bigcup_{j=1}^{l} \pi^{-1} U_j$ does not exceed one. There exists a system $\alpha = \{A_1, \dots, A_m\}$ of multiplicity ≤ 3, consisting of open subsets of the sphere S^2 and covering the compactum $S^2 \setminus \bigcup_{j=1}^{l} U_j$, such that the system

$$\pi^{-1}(\alpha) = \{\pi^{-1} A_s \mid s = 1, \dots, m\}$$

is inscribed in the cover ω and has multiplicity ≤ 2 on the complement of the set $B \setminus \bigcup_{j=1}^{l} \pi^{-1} U_j$. Moreover, one may assume that the body of the system $\pi^{-1}(\alpha)$ is contained in the set G, where the multiplicity of the system β is not greater than one. Then the system $\beta \cup \pi^{-1}(\alpha)$ will be a cover of the bicompactum B, inscribed in the cover ω and having multiplicity ≤ 3.

The bicompactum B is not perfectly normal. It contains a subset of cardinality continuum which is discrete in the induced topology. The question remains open as to whether the dimensions dim and ind coincide for perfectly normal bicompacta.

§ 3. **General construction.** Let X be an arbitrary bicompactum, and let to each point $x \in X$ there be assigned a certain bicompactum Y_x. Suppose that for each point $x \in X$ a continuous mapping $h_x : X \setminus \{x\} \to Y_x$ is defined. Then the system $\{X, Y_x, h_x \mid x \in X\}$ defines in the following way a bicompactum B, continuously mapped onto the bicompactum X by means of a projection $\pi : B \to X$ such that $\pi^{-1}(x) = Y_x$ for all $x \in X$: the bicompactum B is defined on the set $\bigcup_{x \in X} Y_x$; a base for the topology is formed by all possible sets of the form

$$O(U, x, W) = W \cup \pi^{-1}(U \cap h_x^{-1} W),$$

where W is an open subset of the bicompactum Y_x, and U is a neighborhood of the point x in the bicompactum X. We shall denote the bicompactum B by $B\{X, Y_x, h_x\}$. In what follows we shall always assume that the functions h_x satisfy the following property: for every nonempty open set $W \subset Y_x$ and every neighborhood U of the point x, the set $U \cap h_x^{-1} W$ contains a closed set F which partitions the space X into such open sets D and E that $x \in D \subset U$.

Theorem 2. If $\operatorname{Ind} X \leq n$ and $\dim Y_x \leq n$ for all $x \in X$, then

$$\dim B\{X, Y_x, h_x\} \leq n.$$

Theorem 3. If $\dim X \geq n$ and $\operatorname{ind} Y_x \geq n$ for all $x \in X$, then

$$\operatorname{ind} B\{X, Y_x, h_x\} \geq 2n - 1.$$

Theorem 4. If the bicompactum X is hereditarily normal, $\operatorname{Ind} X \leq n$, and $\operatorname{ind} Y_x \leq n$ for all $x \in X$, then

$$\operatorname{ind} B\{X, Y_x, h_x\} \leq 2n.$$

Corollary. If the bicompactum X is hereditarily normal, $\dim X = \operatorname{Ind} X = n$, and $\operatorname{ind} Y_x = n$ for all $x \in X$, then

$$\dim B \leq n \quad \text{and} \quad 2n - 1 \leq \operatorname{ind} B \leq 2n.$$

Remark. There exist families $\{X, Y_x, h_x\}$ satisfying all the conditions of this paragraph. Indeed, one such family is the following: the space X coincides with the n-dimensional sphere S^n, the spaces Y_x for all $x \in X$ coincide with the n-dimensional torus T^n (instead of T^n one may take an arbitrary n-dimensional connected and locally connected compactum), and the mappings $h_x : S^n \setminus \{x\} \to$

T^n are obtained from a mapping $g : J \to T^n$ in the same way as, in the first paragraph, the mappings $h_x : S^2 \setminus \{x\} \to T^2$ are obtained from the mapping $g : J \to T^2$.

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Note: Figure translations are in progress. See original paper for figures.