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Abstract

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MATHEMATICS

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A VARIATIONAL METHOD FOR SOLVING IRREGULAR ELLIPTIC EQUATIONS

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We study the equation

$$Au = (-1)^l \sum_{\substack{|\alpha|=l \\ |\beta|=l}} D^\beta[a_{\alpha\beta}(x) D^\alpha u(x)] = f$$

in an arbitrary open set O of the n-dimensional Euclidean space E_x^n of variables $x = (x_1, \dots, x_n)$, where α, β are n-dimensional vectors with integer nonnegative coordinates, and $|\alpha|, |\beta|$ are the sums of these coordinates. As the domain of definition $\Omega(A)$ of the operator A we take finite real-valued functions $u = u(x)$ having generalized square-summable derivatives $D^\alpha u(x)$ of order l such that the functions $a_{\alpha\beta}(x) D^\alpha u(x)$, in turn, have square-summable generalized derivatives $D^\beta[a_{\alpha\beta}(x) D^\alpha u(x)]$, $|\beta| = l$. The operator A is assumed to be nonnegative and symmetric,

$$(Au, u) = \int_O Au \cdot u \, dx \geq 0, \qquad (Au, v) = (u, Av)$$

for all $u, v \in \Omega(A)$.

Let H be the Hilbert space that is the completion of $\Omega(A)$ in the metric defined by the scalar product $[u, v] = (Au, v)$, and let H^* be its dual.

A generalized solution of the equation $Au = f$, $f \in H^*$, is a function $u_0 \in H$ such that $[u_0, v] = f(v)$ for all $v \in H$. The variational method of solving the equation $Au = f$ consists in finding an element of the space H that realizes the minimum of the functional $F(u) = [u, u] - 2f(u)$. As is known, the generalized solution always exists and is unique in H, and can be found by the variational method.

Let $a = a(x)$ be a measurable almost everywhere positive function,

$$D^l u = \left[\sum_{i_1,\dots,i_l=1}^{n} \left[\frac{\partial^l u(x)}{\partial x_{i_1} \dots \partial x_{i_l}} \right]^2 \right]^{1/2},$$

$1 < p < \infty$. We define the space $L^{(l)}_{p,a}(O)$ as the completion of finite functions having generalized derivatives of order l, in the norm

$$\|u, L^{(l)}_{p,a}(O)\| = \left[\int_O a(x)|D^l u|^p \, dx \right]^{1/p}.$$

In the case $l = 0$ we shall write $L^{(0)}_{p,a}(O) \equiv L_{p,a}(O)$.

The operator A is called elliptic if there exists a measurable almost everywhere positive function $a(x)$ such that, for all $u \in \Omega(A)$,

$$(Au, u) \geq \int_O a(x)|D^l u|^2 \, dx.$$

Thanks to the embedding $H \subset L^{(l)}_{2,a}(O)$, for an elliptic operator A the investigation of the solution—its differential properties, satisfaction of boundary conditions, etc.—is reduced to the study of the space $L^{(l)}_{2,a}(O)$.

In the proof of theorems of the type $L^{(l)}_{p,a}(O) \subset L_{p,b}(O)$, generalizations of the known Hardy inequalities are used. Let $u(x)$ and $a(x)$ be nonnegative measurable functions on the real axis, M and N numbers, $-\infty \leq M < N \leq +\infty$, and l a natural number. Introduce the notation

$$\langle l \rangle \int_M^x u(t)\, dt = \int_M^x dt_{l-1} \int_M^{t_{l-1}} dt_{l-2} \cdots \int_M^{t_1} u(t)\, dt.$$

Similarly, $\langle l \rangle \displaystyle\int_x^N u(t)\, dt$ is defined.

Lemma. The inequalities

$$\int_M^N (N-x)^{(l-1)p/(p-1)} \left[\int_M^x (N-t)^{(l-1)p/(p-1)} a^{1/(1-p)}(t)\, dt \right]^{-p} \left[\langle l \rangle \int_M^x u(t)\, dt \right]^p dx \leq$$

$$\leq \frac{1}{[(l-1)!]^p} \left(\frac{p}{p-1} \right)^p \int_M^N a(x) u^p(x)\, dx, \qquad -\infty \leq M < N < +\infty;$$

$$\int_M^N (x-M)^{-(l-1)p} a^{1/(1-p)}(x) \left[\int_M^x a^{1/(1-p)}(t)\,dt\right]^{-p} \left[\langle l\rangle \int_M^x u(t)\,dt\right]^p dx \leq$$

$$\leq \frac{1}{[(l-1)!]^p}\left(\frac{p}{p-1}\right)^p \int_M^N a(x)u^p(x)\,dx, \qquad -\infty < M < N \leq +\infty,$$

and the analogous ones for estimating $\langle l\rangle \int_x^N u(t)\,dx$. For $l=1$ both limits of integration may be infinite.

Other generalizations of Hardy inequalities were obtained by V. R. Portnov (^{1}) and F. A. Sysoeva (^{2}).

Suppose that in the open set O a system of coordinates is introduced which maps O one-to-one onto an open set $\widetilde{O}$ of the n-dimensional Euclidean space of variables (r,γ), where r is the distinguished variable and γ is an $(n-1)$-dimensional vector. The functions $x_i(r,\gamma)$, $i=1,2,\dots,n$, will be assumed l times continuously differentiable in the closure of the set $\widetilde{O}$, and the Jacobian $D(x)/D(r,\gamma)$ positive in $\widetilde{O}$.

Passing in the integral $\int_O a(x)|D^l u|^p dx$ to the variables (r,γ) and using the generalized Hardy inequalities, one can always find a weight $b(x)$ such that

$$\int_O b(x)|u(x)|^p dx \leq \int_O a(x)|D^l u|^p dx.$$

With a sufficiently strong degeneration of the weight $a(x)$, the elements of $L^{(l)}_{p,a}(O)$, generally speaking, need not be locally summable functions. The weight $b(x)$ depends essentially on the geometric properties of the set in every neighborhood of whose points the function $a^{1/(1-p)}(x)$ is nonintegrable. For example, one can give an example of a weight $a(x)$ for which $a^{1/(1-p)}(x)$ is nonintegrable in a neighborhood of the points of the surface of a ball lying in O, and there exists a function belonging to $L^{(l)}_{p,a}(O)$ and identically equal to $+\infty$ inside the ball. The corresponding weight $b(x)$ inside the ball is equal to zero.

We investigate the question of preservation of boundary conditions for functions from the class $L^{(l)}_{p,a}(O)$. Boundary values are understood in the sense of the limit of the function almost everywhere along the given field of directions r. Moreover, if one is speaking of boundary values of derivatives of the function, derivatives with respect to r are meant. This definition is applicable to a set O such that $\operatorname{mes}\Pi_r\partial\widetilde{O} > 0$, where $\Pi_r E$ denotes the projection of the set $E \subset E^n_{(r,\gamma)}$ onto the hythe hyperplane Γ of the vectors γ, and $\partial\widetilde{O}$ is the boundary of $\widetilde{O}$. We shall assume that the indicated condition is satisfied for each connected component of the set O. For questions connected with boundary conditions at infinity for unbounded domains, including the whole of E^n, see (3).

In a certain sense one may also speak of boundary values in the mean.

Let us also denote by $\overline{\partial}G(\underline{\partial}G)$ the upper (lower) semiboundary of an open set $G \subset E^n_{(r,\gamma)}$, i.e., those boundary points which are the upper (lower) endpoints of the intervals obtained by intersecting G with the straight lines $\gamma = \text{const}$.

Theorem 1. *Let there exist a finite or countable system* $\{O_i\}$ *of open sets* $O_i \subset O$ *such that*

$$\operatorname{mes} \Pi_\Gamma \left(\partial\widetilde{O} \setminus \bigcup_i \partial\widetilde{O}_i \right) = 0$$

and

$$\operatorname{mes} \Pi_\Gamma \left(\partial\widetilde{O} \setminus \bigcup_i \partial\widetilde{O}_i \right) = 0,$$

and for each O_i

$$\int_{O_i} \left[\frac{D(x)}{D(r,\gamma)} \right]^{-p/(p-1)} a^{1/(1-p)}(x)\, dx < \infty,$$

while the field of directions satisfies the condition $\partial^2 x/\partial r^2 = 0$. *Then every function* $u(x) \in L^{(l)}_{p,a}(O)$, *together with its derivatives up to order* $l-1$ *inclusive, is equal to zero on the boundary of* O *both in the mean and in the sense of almost everywhere.*

Let us note that here $u(x)$ need not be a locally summable function not only inside O, but also in an arbitrarily small neighborhood of its boundary.

Theorem 2. *Let the inequality*

$$\int \left[\frac{D(x)}{D(r,\gamma)} \right]^{-p/(p-1)} a^{1/(1-p)}(x)\, dx < \infty$$

be satisfied for all of O *or for any finite part of it, and let the condition on the field of directions be as before. Then the function* $u(x) \in L^{(l)}_{p,a}(O)$ *has all generalized derivatives up to order* l *inclusive. If, moreover,*

$$a^{1/(1-p)}(x) \in L^{\text{loc}}_s(O)$$

for some s, $1 \le s \le \infty$, *then the generalized derivatives of order* l *belong to*

$$L^{\text{loc}}_{\frac{p}{1+(p-1)/s}}(O).$$

The requirement $\partial^2 x/\partial r^2 = 0$ is inessential and was introduced for brevity of the formulas. For a general field of directions we restrict ourselves to $l = 1$, bearing in mind that one can use embedding theorems of the type

$$L^{(l)}_{p,a}(O) \subset L^{(k)}_{p,b_k}(O).$$

Theorem 3. *Let, in O or in any finite part of it,*

$$\int \left[\frac{D(x)}{D(r,\gamma)}\right]^{-p/(p-1)} a^{1/(1-p)}(x)\,dx < \infty.$$

Then the function $u(x) \in L^{(l)}_{p,a}(O)$ has first generalized derivatives and assumes on the boundary the value zero almost everywhere and in the mean. If, moreover,

$$a^{1/(1-p)}(x) \in L^{\text{loc}}_{s}(O)$$

for some s, $1 \le s \le \infty$, then the first generalized derivatives belong to

$$L^{\text{loc}}_{\frac{p}{1+(p-1)/s}}(O).$$

Let us return to the equation $Au = f$.

Theorem 4. Suppose that the conditions of Theorem 2 or 3 are satisfied with $p = 2$. Then, for the elliptic equation $Au = f$, there exists a generalized solution, unique up to a set of measure zero, for any right-hand side $f \in H^*$. This solution belongs to the space $L^{(l)}_{2,a}(O)$, has generalized derivatives up to order l inclusive, and satisfies the zero boundary conditions almost everywhere and, on the average, together with its derivatives up to order $l-1$ inclusive. If, in addition, the coefficients of the equation

$$a_{\alpha\beta}(x) \in L^{\text{loc}}_{\frac{2}{1-1/s}}(O)$$

(where s is the same as in the conditions of Theorem 2 or 3), then the equation is satisfied in the following sense: $a_{\alpha\beta}(x)D^\alpha u(x)$ are locally summable functions and have, in the sense of the theory of generalized functions, derivatives $D^\beta(a_{\alpha\beta}(x)D^\alpha u(x))$, the sum of which is equal to f.

We now consider the equation $Au = f$ with right-hand side a function $f = f(x) \in L_{2,1/b}(O)$. It is assumed that $b(x) > 0$ almost everywhere. The embedding $L_{2,1/b}(O) \subset H^*$ holds.

Theorem 5. For any function $f(x) \in L_{2,1/b}(O)$, the elliptic equation $Au = f$ has, and moreover uniquely, a generalized solution in the space $L^{(l)}_{2,a}(O)$. The resulting extension of the operator

$$\widetilde{A} = \frac{1}{b(x)}A,$$

acting from $L_{2,b}(O)$ to $L_{2,b}(O)$, is a self-adjoint operator with positive (equal to one) lower bound.

Under stronger restrictions on the coefficients and the domain, similar questions were also studied earlier (see, for example, ([4]–[8])). For unbounded domains, the variational method was extended by L. D. Kudryavtsev ([9], [10]); his results were developed by V. R. Portnov ([1], [11]), Yu. S. Nikol' skii ([12], [13]), and T. S. Pitolkina ([14]).

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Note: Figure translations are in progress. See original paper for figures.