GENERALIZED SELF-ADJOINT EXTENSIONS OF SYMMETRIC OPERATORS

MATHEMATICS

1968

SovietRxiv

Abstract

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UDC 513.88+517.948

MATHEMATICS

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GENERALIZED SELF-ADJOINT EXTENSIONS OF SYMMETRIC OPERATORS

(Presented by Academician L. S. Pontryagin on 3 V 1967)

In the present paper a new kind of extension of symmetric operators is introduced and studied. It turns out that any symmetric operator A can always be extended to the domain of definition of A^* in such a way that the extension obtained is symmetric with respect to a certain generalized "scalar product." A number of properties of such extensions are established, and their classification and description are given. This makes it possible to characterize (for operators with finite and equal deficiency indices) all ordinary symmetric and self-adjoint extensions in terms of abstract boundary conditions, where the boundary conditions are a certain combination of generalized elements (functionals) "orthogonal" to the domain of definition of the symmetric operator*.

Let G_0 be a certain complete Hilbert space with scalar product $(f, g)_0$. Suppose that in G_0 there is an everywhere dense linear set G_+, which is a complete Hilbert space with respect to another scalar product $(f, g)_+$. We shall assume that $\|f\|_0 \leq \|f\|_+$ $(f \in G_+)$. The space G_+, as is known ([1]), is called a space with positive norm, and also a space of basic elements. We shall say that every antilinear functional $a(f)$ on G_+ is generated by a generalized element a, and we shall write $a(f) \equiv (a, f)_0$ $(f \in G_+)$. In what follows we shall use the notation $\overline{(a, f)_0} = (f, a)_0$. Obviously, the totality G_- of all generalized elements is a linear set.

It is easy to see that

$$(a, f)_0 = (Ja, f)_0 \qquad (f \in G_+,\ a \in G_-). \tag{1}$$

Equality (1) generates a linear operator J mapping the set G_- onto the space G_+. Introduce in G_- a scalar product by setting

$$(\alpha, \beta)_- = (J\alpha, J\beta)_+ \qquad (\alpha, \beta \in G_-).$$

The operator J, obviously, is an isometric operator mapping G_- onto G_+. Thus, $G_+ \subseteq G_0 \subseteq G_-$. The space G_- will be called a space with negative norm, and also the space of generalized elements of the Hilbert space G_0. It is not difficult to show ([1]) that G_+ is dense in G_-.

Let $\Omega(f,g)$ be a bilinear functional in G_+. Then, as is known ([2,3]),

$$\Omega(f,g) = (Bf,g)_0 \qquad (f,g \in G_+), \tag{2}$$

where B is a bounded linear operator acting from G_+ into G_-, which is uniquely determined by the functional Ω.

Now let B be an arbitrary bounded linear operator acting from G_+ into G_-. The expression $(f,Bg)_0$ $(f,g \in G_+)$ is, obviously, a bilinear functional in G_+. Then, according to (2), uniquely

* In a somewhat different form, abstract boundary conditions occur in ([4]).

there exists a bounded linear operator $B^\times$, also mapping G_+ into G_-, for which

$$(f,Bg)_0 = (B^\times f,g)_0 \qquad (f,g \in G_+).$$

We shall call the operator $B^\times$ the **generalized adjoint** operator with respect to B. If $B = B^\times$, then such an operator will be called a **generalized self-adjoint** operator.

Let A be a symmetric operator with dense domain, acting in the Hilbert space G_0. In the linear set $G_+ = D_{A^*}$ introduce the scalar product

$$(f,g)_+ = (A^*f, A^*g)_0 + (f,g)_0 \qquad (f,g \in G_+). \tag{3}$$

Since A^* is closed, G_+ is a complete Hilbert space. Construct the triple of spaces $G_+ \subseteq G_0 \subseteq G_-$. According to von Neumann' s formulas,

$$D_{A^*} = D_A + \mathfrak{N}_\lambda + \mathfrak{N}_{\bar\lambda}.$$

Theorem 1. *In order that the subspaces $D_A, \mathfrak{N}_\lambda, \mathfrak{N}_{\bar\lambda}$ be pairwise orthogonal in G_+, it is sufficient that $\lambda = \pm i$.*

In what follows we shall use the representation of G_+ in the form

$$G_+ = D_A \oplus \mathfrak{N}_i \oplus \mathfrak{N}_{-i}. \tag{4}$$

Denote $\mathfrak{M} = \mathfrak{N}_i \oplus \mathfrak{N}_{-i}$. Every vector $f \in D_{A^*}$, by virtue of (4), is uniquely represented in the form

$$f = f_A + f_{\mathfrak{N}} \qquad (f_A \in D_A, \quad f_{\mathfrak{N}} \in \mathfrak{M}).$$

Theorem 2. *In order that the symmetric operator A can be extended to the space $G_+ = D_{A^*}$ so that the resulting extension A_{G_+} $(G_+ \to G_-)$ is a generalized self-adjoint operator, it is necessary and sufficient that there exist a linear operator $P(\mathfrak{M} \to G_-)$ for which*

$$(Pf_{\mathfrak{N}}, g_A)_0 = (f_{\mathfrak{N}}, Ag_A)_0, \qquad (Pf_{\mathfrak{N}}, g_{\mathfrak{N}})_0 = (f_{\mathfrak{N}}, Pg_{\mathfrak{N}})_0$$

$$(f_{\mathfrak{N}}, g_{\mathfrak{N}} \in \mathfrak{M}, \quad g_A \in D_A).$$

In this case

$$A_{G_+} f = Af_A + Pf_{\mathfrak{N}} \qquad (f = f_A + f_{\mathfrak{N}},\ f_A \in D_A,\ f_{\mathfrak{N}} \in \mathfrak{M}).$$

Relying on Theorem 2, Theorem 3 is proved.

Theorem 3. *Every closed symmetric operator A with dense domain can be extended to the whole space $G_+ = D_{A^*}$ so that the resulting extension A_{G_+} is a generalized self-adjoint operator acting from G_+ into G_-.*

Let now the operator A have defect index (r, r) $(r < \infty)$. Then, according to (4),

$$f = f_A + \sum_{j=1}^{r} \xi_j e_j + \sum_{j=1}^{r} \eta_j g_j,$$

where $\{e_j\}_1^r, \{g_j\}_1^r$ are orthonormal bases in the subspaces $\mathfrak{N}_i, \mathfrak{N}_{-i}$. Introduce the notation

$$\hat{e}_j = J^{-1} e_j, \qquad \hat{g}_j = J^{-1} g_j \quad (j = 1, 2, \dots, r).$$

It is not difficult to verify that $\hat{e}_j, \hat{g}_j$ $(j = 1, 2, \dots, r)$ are generalized elements of the Hilbert space G_0 and are "orthogonal" to D_A, i.e.

$$(\hat{e}_j, f_A)_0 = (\hat{g}_j, f_A)_0 = 0 \qquad (f_A \in D_A).$$

Theorem 4. *Let A be a symmetric operator with defect index (r, r) $(r < \infty)$, acting in the Hilbert space G_0. In order*

in order that the expression

$$A_{G_+}f = A^*f + \sum_{j,k=1}^{r} [a_{jk}(f,\hat{e}_j)_0 + b_{jk}(f,\hat{g}_j)_0]\hat{g}_k + \tag{5}$$

$$+ \sum_{j,k=1}^{r} [c_{jk}(f,\hat{e}_j)_0 + d_{jk}(f,\hat{g}_j)_0]\hat{e}_k$$

be a generalized self-adjoint extension of the operator A on $G_+ = D_{A^*}$, it is necessary and sufficient that the coefficient matrices satisfy the relations

$$D = A^*, \qquad c_{nm} = \overline{c_{mn}}, \qquad b_{nm} = \overline{b_{mn}} \quad (m \neq n), \tag{6}$$

$$\operatorname{Im} c_{nn} = -\frac{1}{2}, \qquad \operatorname{Im} b_{nn} = \frac{1}{2}.$$

On the basis of Theorem 2, one can give a complete description of all generalized self-adjoint extensions of a symmetric operator with equal finite defect numbers.

Theorem 5. Every generalized self-adjoint extension A_{G_+} of a symmetric operator A on $G_+ = D_{A^*}$ has the form (5), and the coefficient matrices satisfy the relations (6).

Definition. A generalized self-adjoint extension A_{G_+} of the operator A on $G_+ = D_{A^*}$ will be called **weak** if A_{G_+} is not an extension of any symmetric extension $\tilde{A}$ of the operator A.

Theorem 6. In order that the expression (5) with relations (6) be a weak extension A_{G_+} of the operator A on $G_+ = D_{A^*}$, it is necessary and sufficient that the matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be nonsingular.

It follows from Theorems 5 and 6 that every symmetric operator with defect index (r, r) $(r < \infty)$ always has a weak generalized self-adjoint extension. The totality of all such extensions is described by expression (5) with conditions (6), where the matrix T is nonsingular.

Theorems analogous to Theorems 4, 5, and 6 can also be obtained for operators with finite but unequal defect numbers.

Theorem 7. In order that the expression (5), when the conditions (6) are fulfilled, be also an extension of some symmetric extension $\tilde{A}$ $(\tilde{A}^* \neq \tilde{A})$ of the operator A, it is necessary and sufficient that

$$\operatorname{rang}\begin{pmatrix} A & B \\ C & D \end{pmatrix} < r. \tag{7}$$

It follows from this theorem that the domain of definition of an ordinary symmetric extension $\tilde{A}$ of the operator A consists of vectors $f \in G_+ = D_{A^*}$ for which

$$\sum_{j=1}^{r} [a_{jk}(f, \hat{e}_j)_0 + b_{jk}(f, \hat{g}_j)_0] = 0,$$

$$(k = 1, 2, \dots, r)$$

$$\sum_{j=1}^{r} [c_{jk}(f, \hat{e}_j)_0 + d_{jk}(f, \hat{g}_j)_0] = 0$$

for some coefficient matrices satisfying the conditions (6) and (7).

Theorem 8. In order that the expression (5), when the conditions (6) are fulfilled, be an extension of some self-adjoint extension $\tilde{A}$

of the operator A, it is necessary and sufficient that

$$\operatorname{rang}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = r. \tag{8}$$

Theorem 8 can be formulated somewhat differently, namely:

Theorem 9. In order that expression (5), under the conditions (6), be an extension of some self-adjoint extension $\widetilde{A}$ of the operator A, it is necessary and sufficient that there exist a unitary matrix U such that

$$A + UB = 0, \qquad C + UD = 0. \tag{9}$$

The extensions considered in Theorems 7 and 8 will be called, respectively, **medium** and **strong**. From Theorems 8 and 9 there follows a curious result for matrices.

In order that the matrix (8) with conditions (6) have rank r, it is necessary and sufficient that the conditions (9) be satisfied.

I express my gratitude to V. I. Matsayev for a number of valuable suggestions and for discussion of the present work.

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Received
26 IV 1967

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Note: Figure translations are in progress. See original paper for figures.