ON SOLVABLE GROUPS OF FINITE RANK

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A rational series of a group $\mathfrak{G}$ is an ascending normal series of this group

$$\mathfrak{R}_0 = E \subset \mathfrak{R}_1 \subset \ldots \subset \mathfrak{R}_\alpha \subset \mathfrak{R}_{\alpha+1} \subset \ldots \subset \mathfrak{R}_\gamma = \mathfrak{G},$$

whose factors $\mathfrak{R}_{\alpha+1}/\mathfrak{R}_\alpha$ are isomorphic to certain subgroups of the additive group of rational numbers ([1]).

For a group $\mathfrak{G}$ possessing a rational series, the notion of rational rank is introduced. Namely, if $\mathfrak{G}$ has a rational series of finite length k, then the rational rank of $\mathfrak{G}$ is taken to be k; otherwise the rational rank of the group is considered infinite. We shall denote the rational rank of the group $\mathfrak{G}$ by $r(\mathfrak{G})$. The symbol $s(\mathfrak{G})$ denotes the special rank ([2]) of the group $\mathfrak{G}$.

The question naturally arises of the relation between $r(\mathfrak{G})$ and $s(\mathfrak{G})$. In this direction V. M. Glushkov ([3]) established that in the case of a locally nilpotent group $\mathfrak{G}$ the equality $r(\mathfrak{G}) = s(\mathfrak{G})$ holds. In the present article it is reported (Theorem 2) that, for any group possessing a rational series, the special rank is equal to the rational rank.

The proof of the main result is based on a number of properties of solvable groups satisfying the weak minimal condition for subgroups. Let us recall the definition.

A group $\mathfrak{G}$ satisfies the weak minimal condition for subgroups if there does not exist in it an infinite descending chain of subgroups

$$\mathfrak{G}_1 \supset \mathfrak{G}_2 \supset \ldots \supset \mathfrak{G}_k \supset \mathfrak{G}_{k+1} \supset \ldots,$$

satisfying the following condition: the index $[\mathfrak{G}_k : \mathfrak{G}_{k+1}]$ of the subgroup $\mathfrak{G}_{k+1}$ in the subgroup $\mathfrak{G}_k$ is infinite, $k = 1, 2, \ldots$. Groups satisfying this minimal condition were studied by the author in ([4]).

We now formulate an auxiliary proposition, which gives a condition that singles out, from the class of nilpotent groups of finite rank, the class of nilpotent groups satisfying the weak minimal condition for subgroups.

Lemma 1. *Let there exist in the nilpotent group of finite rank* $\mathfrak{G}$ *a system of generating elements* $X_1, X_2, \dots, X_k$, $\mathfrak{M}$, *and such a natural number* n *that for every element* $Y \in \mathfrak{M}$ *there is an* l *for which*

$$Y^{n^l} \in \{X_1, X_2, \dots, X_k\}.$$

Then $\mathfrak{G}$ *satisfies the weak minimal condition for subgroups, and for every* $G \in \mathfrak{G}$ *one can specify a number* m *for which*

$$G^{n^m} \in \{X_1, X_2, \dots, X_k\}.$$

Conversely, if a nilpotent group $\mathfrak{G}$ *satisfies the weak minimal condition for subgroups, then in* $\mathfrak{G}$ *there exists such a finite system of elements* $X_1, X_2, \dots, X_k$ *and such a natural number* n *that for every element* $G \in \mathfrak{G}$ *one can specify a number* m *for which* $G^{n^m} \in \{X_1, X_2, \dots, X_k\}$.

Corollary 1. *In a nilpotent group of finite rank* $\mathfrak{G}$, *a subgroup generated by a finite set of subgroups satisfying the weak minimal condition also satisfies the weak minimal condition.*

Corollary 2. *In a locally nilpotent torsion-free group* $\mathfrak{G}$, *a subgroup generated by a finite set of subgroups satisfying the weak minimal condition also satisfies the weak minimal condition.*

This follows from Theorem 4 of paper ([3]) and Corollary 1 of Lemma 1. Let us note that for a periodic locally nilpotent group, Corollary 2 does not hold. A counterexample can be found in the paper of O. Yu. Schmidt ([5]).

One characterization of nilpotent torsion-free groups of finite rank is given by

Lemma 2. *A nilpotent torsion-free group* $\mathfrak{R}$ *has finite rank if and only if there does not exist in* $\mathfrak{R}$ *an infinite system of elements* $X_1, X_2, \dots, X_i, \dots$ *having the following property:*

$$\{X_1, X_2, \dots, X_i\} \cap \{X_{i+1}\} = E, \qquad i = 1, 2, \dots .$$

Lemma 3. *Let* φ *be an automorphism of a nilpotent group of finite rank* $\mathfrak{R}$, *and let* $\mathfrak{L}$ *be a subgroup of* $\mathfrak{R}$ *satisfying the weak minimality condition for subgroups. Then the least subgroup* $\mathfrak{L}^*$, *invariant with respect to* φ *and containing* $\mathfrak{L}$, *also satisfies the weak minimality condition for subgroups.*

Lemma 4. *If* $\mathfrak{G}$ *is a nilpotent group of finite rank and the factor group* $\mathfrak{G}/\mathfrak{G}'$ *of the group* $\mathfrak{G}$ *by its commutator subgroup* $\mathfrak{G}'$ *satisfies the weak minimality condition for subgroups, then* $\mathfrak{G}$ *also satisfies the weak minimality condition for subgroups.*

Relying on the formulated lemmas and the results of the work of A. I. Mal' cev ([6]), one can establish the following theorem, which is of definite interest for the theory of solvable groups of finite rank.

Theorem 1. *A solvable torsion-free group of finite rank with a finite number of generators satisfies the weak minimality condition for subgroups.*

Lemma 5. *Let a group $\mathfrak{G}$ possess a rational series of length k and satisfy the weak minimality condition for subgroups. Then in $\mathfrak{G}$, for every sufficiently large prime number p, there exist two characteristic subgroups of finite index $\mathfrak{A}, \mathfrak{B}$, such that $\mathfrak{A} \supset \mathfrak{B}$ and the factor group $\mathfrak{A}/\mathfrak{B}$ is an elementary abelian p-group of order p^k.*

From Lemma 5 and Theorem 1 there immediately follows the main result of the present work.

Theorem 2. *If $\mathfrak{G}$ is a group possessing a rational series, then the special rank of $\mathfrak{G}$ is equal to its rational rank, i.e. $s(\mathfrak{G}) = r(\mathfrak{G})$.*

We indicate some known results obtained as consequences of Theorem 2.

Corollary 1. (V. M. Glushkov ([3])). *In order that a locally nilpotent torsion-free group be a nilpotent group of finite special rank k, it is necessary and sufficient that it possess a rational series of length k.*

Corollary 2. (S. N. Chernikov ([1]), B. I. Plotkin ([7])). *If a group possesses a finite rational series, then the length of such a series is an invariant of the group. The length of a rational series of a proper isolated subgroup is strictly less than the length of a rational series of the whole group.*

Corollary 3. (B. I. Plotkin ([8])). *Let a group $\mathfrak{G}$ possess an ascending rational series. In this case, in order that $\mathfrak{G}$ have finite rank, it is necessary and sufficient that $\mathfrak{G}$ possess finite special rank.*

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