SOLUTION OF THE GRAVITATIONAL EQUATIONS IN A HOMOGENEOUS, COMPLETELY ANISOTROPIC MODEL

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SOLUTION OF THE GRAVITATIONAL EQUATIONS IN A HOMOGENEOUS, COMPLETELY ANISOTROPIC MODEL

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In this article, solutions of the equations of the general theory of relativity are considered for the case of a homogeneous but completely anisotropic 3-space in the presence of matter. These anisotropic solutions (with some exceptions) asymptotically tend to the isotropic Friedmann solution in the flat model; however, the anisotropy manifests itself essentially near singularities. A special case of solutions of the type considered is the flat model with axial symmetry ([2–4]).

A completely anisotropic solution for dust-like matter was given in ([1]). Below, the corresponding formulas are given for an ultrarelativistic equation of state.

1. As the coordinate system, a comoving system is used, in which the medium is at rest. This comoving system is synchronous (so that the lines of proper time τ, coinciding with the world lines of the "fluid" particles, are geodesics of 4-space). The metric in the model under consideration has the form

$$-ds^2 = -(c\,d\tau)^2 + [R_1(\tau)dx^1]^2 + [R_2(\tau)dx^2]^2 + [R_3(\tau)dx^3]^2. \quad (1)$$

The functions $R_1(\tau), R_2(\tau), R_3(\tau)$ are determined by the Einstein gravitational equations (the notation of ([5]) is used)

$$R_i^k = (8\pi k/c^4)\,[T_i^k - (T/2)\delta_i^k]. \quad (2)$$

In the metric (1) the 3-space x^1, x^2, x^3 is Euclidean; however R_i^k differ from 0, and the 4-space is curved.

In equations (2), T_i^k is the energy-momentum tensor of an ideal gas. In the comoving system used, T_i^k has diagonal form: $T_1^1 = T_2^2 = T_3^3 = p$, where p is the pressure; $-T_0^0 = e$, where e is the density of internal energy; $T = 3p - e$; p and e depend on τ.

The determinant of the metric tensor $(-g)$ is equal to

$$(-g)^{1/2} = R_1(\tau)R_2(\tau)R_3(\tau). \tag{3}$$

The nonzero components of equations (2) have the form ([5]) (a dot denotes differentiation with respect to τ):

$$R_0^0: \ \frac{1}{c^2}\frac{d^2}{d\tau^2}\ln\sqrt{-g} + \frac{1}{c^2}\left(\frac{\dot R_1^2}{R_1^2} + \frac{\dot R_2^2}{R_2^2} + \frac{\dot R_3^2}{R_3^2}\right) = -\frac{8\pi k}{c^4}\frac{e+3p}{2}, \tag{4}$$

$$R_1^1,\ R_2^2,\ R_3^3: \quad \frac{1}{c^2\sqrt{-g}}\frac{d}{d\tau}\left[\sqrt{-g}\frac{d}{d\tau}\ln R_\alpha\right] = \frac{8\pi k}{c^4}\frac{e-p}{2}, \quad \alpha = 1,2,3. \tag{5}$$

The equations $T^k_{i;k} = 0$ contained in (2) give ($i = 0$)

$$-\dot e/(\dot e + p) = (-\dot g)/2(-g). \tag{6}$$

Adding the three equations (5), we obtain an equation for $(-g)$

$$d^2[(-g)^{1/2}]/d\tau^2 = (12\pi k/c^2)(e-p)(-g)^{1/2}. \tag{7}$$

Equations (6) and (7) are conveniently used to determine, for a given equation of state, the dependence $-g(\tau)$, after which R_1, R_2, and R_3 are determined from (4) and (5).

In the model (1) under consideration, the three Hubble "constants" $h_\beta = d\ln R_\beta/d\tau$, $\beta = 1,2,3$, satisfy the relation obtained after substituting h_β into the equation $R_0^0 - (R'/2) = (8\pi k/c^4)T_0^0$

$$h_1h_2 + h_1h_3 + h_2h_3 = 8\pi ke/c^2. \tag{8}$$

2. Dust-like matter ($p = 0,\ e = \mu c^2,\ \mu$ is the mass density)

From (6) it follows that $e = \text{const}/(-g)^{1/2}$ (conservation of mass in a "liquid" volume), and from (7) it follows that

$$(-g)^{1/2} = Lc^2(\tau-\tau_1)(\tau-\tau_2), \qquad L = \text{const}, \quad \tau_1 = \text{const}, \quad \tau_2 = \text{const}. \tag{9}$$

The formal possibility of complex conjugate values τ_1 and τ_2 in (9) does not lead to real values for R_1, R_2, and R_3, and is not physically realized. This is connected with the fact that the determinant $(-g)$ does not vanish, which is impossible in a synchronous system ([5,6]).

From equations (5), in consequence of (9), it follows that

$$\begin{aligned}(R_1)^3 &= L_1 c^2 (\tau-\tau_1)^{1+\alpha_1}(\tau-\tau_2)^{1-\alpha_1},\\ (R_2)^3 &= L_2 c^2 (\tau-\tau_1)^{1+\alpha_2}(\tau-\tau_2)^{1-\alpha_2},\\ (R_3)^3 &= L_3 c^2 (\tau-\tau_1)^{1+\alpha_3}(\tau-\tau_2)^{1-\alpha_3}.\end{aligned} \tag{10}$$

$L_1,\ L_2,\ L_3$ are constants having the meaning of coordinate scales; $L_1 L_2 L_3 = L^3$.

The quantities $\alpha_1, \alpha_2, \alpha_3$ in (10) are constants that are essential in the solution. They are not completely arbitrary, but are connected by two relations. One is a consequence of (9):

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \tag{11}$$

The second relation is obtained after substituting (10) into (4):

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 6. \tag{12}$$

From (11) and (12) it follows that, for a given value of α_1, the values α_2 and α_3 are roots of the equation $\alpha^2 + \alpha\alpha_1 + (\alpha_1^2 - 3) = 0$. These roots are real for $\alpha_1^2 \leq 4$, so that α_1, and likewise α_2 and α_3, do not exceed 2 in absolute value. As α_1 increases monotonically in the interval from -1 to 1, the values α_2 and α_3 decrease monotonically: one from 2 to 1, the other from -1 to -2, together with α_1 covering the whole domain of admissible values. Therefore, without loss of generality, the coordinates may be considered numbered so that

$$-1 \leq \alpha_1 \leq 1, \qquad 2 \geq \alpha_2 \geq 1, \qquad -1 \geq \alpha_3 \geq -2. \tag{13}$$

For $\alpha_1 = -1$ we have $\alpha_3 = -1$ and, according to (10), $R_3 = \text{const}\, R_1$; for $\alpha_1 = 1$ we have $\alpha_2 = 1$ and $R_2 = \text{const}\, R_1$. Both these limiting cases correspond to the flat model with axial symmetry ([2–4]). The values $|\alpha_1| < 1$ correspond to a completely anisotropic solution.

Denoting $\alpha_1 = 2\sin\gamma$, where, according to (13), $-\pi/6 \leq \gamma \leq \pi/6$, we obtain expressions for the constants $\alpha_1, \alpha_2, \alpha_3$ in (10), satisfying (11) and (12), in the form ([1])

$$\alpha_1 = 2\sin\gamma, \qquad \alpha_2 = 2\sin(\gamma + 2\pi/3), \qquad \alpha_3 = 2\sin(\gamma + 4\pi/3), \\ -\pi/6 \leq \gamma \leq \pi/6. \tag{14}$$

From (4) and (5), for the energy density it follows that

$$e = c^2/[6\pi k(\tau-\tau_1)(\tau-\tau_2)]. \tag{15}$$

At the instants of time τ_1 and τ_2 (we take $\tau_1 < \tau_2$), the quantity e becomes ∞, and the solution has singularities. Under complete anisotropy ($|\alpha_1| < 1$), in

at the moment $\tau = \tau_1$, $R_1 = 0,\ R_2 = 0,\ R_3 = \infty$, and at the moment $\tau = \tau_2$, $R_1 = 0,\ R_2 = \infty,\ R_3 = 0$.

Owing to the arbitrariness in the exponents, the solution (10), generally speaking, cannot be analytically continued to both sides of the values $\tau = \tau_1$ and $\tau = \tau_2$. In this connection it should be noted that, along with (10), there are solutions differing from (10) by a change of sign at one or both of the differences $\tau - \tau_1$ and $\tau - \tau_2$. According to (15), for $\tau < \tau_1$ and for $\tau > \tau_2$, $e > 0$; in the interval $\tau_1 < \tau < \tau_2$, $e < 0$. We note that the solution (10) is not a symmetric function of τ (even when $\tau_2 = -\tau_1$). As $|\tau| \to \infty$, the solution (10) becomes isotropic: $R_1 \sim R_2 \sim R_3 \sim \tau^{2/3}$. For $\tau_1 = \tau_2$, (10) corresponds to the flat Friedmann model.

3. The ultrarelativistic equation of state $e = 3p$. From (6) we have $e = \text{const}/(-g)^{2/3}$. We shall consider physically real states with nonnegative values of $e = 3p$, so that

$$e = K/(-g)^{2/3}, \qquad K = \text{const}, \qquad K > 0. \tag{16}$$

Integrating (7) once, we obtain

$$[d(-g)^{1/2}/c\,d\tau]^2 = (24\pi k K/c^4)[(-g)^{1/3} + N], \qquad N = \text{const}. \tag{17}$$

Values $N < 0$ in (17) do not lead to real R_1, R_2, R_3; this is again connected with the fact that for $N < 0$, in the sense of (17), the determinant $(-g)$ does not become 0, which is impossible in a synchronous system.

The value $N = 0$ in (17) corresponds to the isotropic flat Friedmann solution. Values $N > 0$ correspond to an anisotropic solution.

It is convenient to introduce the parameter λ according to

$$(-g)^{1/3} = L^2 \operatorname{sh}^2 2\lambda, \qquad L^2 \equiv N = \text{const}. \tag{18}$$

Equation (17) then takes the form

$$c\,d\tau/d\lambda = 6L^2c^2 \operatorname{sh}^2 2\lambda/(24\pi k K)^{1/2}.$$

Integrating (5), we obtain

$$c\tau/L_0 = \operatorname{sh} 4\lambda - 4\lambda, \qquad L_0 \equiv 3L^2c^2/8(6\pi k K)^{1/2}. \tag{19}$$

The additive constant in (19) is chosen so that for $\lambda = 0$, $\tau = 0$ and $g = 0$.

Integrating (5), we obtain

$$R_1 = L_1(\operatorname{sh}\lambda)^{1+\alpha_1}(\operatorname{ch}\lambda)^{1-\alpha_1}, \qquad R_2 = L_2(\operatorname{sh}\lambda)^{1+\alpha_2}(\operatorname{ch}\lambda)^{1-\alpha_2},$$
$$R_3 = L_3(\operatorname{sh}\lambda)^{1+\alpha_3}(\operatorname{ch}\lambda)^{1-\alpha_3}. \tag{20}$$

L_1, L_2, L_3 are scale constants; $L_1L_2L_3 = 8L^3$.

The quantities $\alpha_1, \alpha_2, \alpha_3$ in (20) are constants for which the relations (11) and (12) again hold, arising as a consequence of (18) and after substitution of (20) into (4). For $\alpha_1, \alpha_2, \alpha_3$, formulas (13) and (14) are valid. The completely anisotropic solution again corresponds to values $|\alpha_1| < 1$.

As in the preceding case, the solution (20), generally speaking, is not analytically continued to both sides of the value $\tau = 0$ ($\lambda = 0$). In this case, for $\lambda < 0$, the solution is obtained from (20) by replacing $\operatorname{sh}\lambda$ with $-\operatorname{sh}\lambda$.

The "Hubble constants" for the solution (20) are equal to

$$h_{1,2,3} \equiv d\ln R_{1,2,3}/d\tau = [(\operatorname{ch}2\lambda + \alpha_{1,2,3})/\operatorname{sh}^3 2\lambda](8\pi kK/3c^2L^4)^{1/2}. \tag{21}$$

At the moment $\tau = 0$ ($\lambda = 0$) the quantity e, according to (16), becomes ∞, and the solution has a singularity.

In the completely anisotropic case ($|\alpha_1| < 1$), at the moment $\tau = 0$, $R_1 = R_2 = 0$, $R_3 = \infty$. Near $\tau = 0$ (for $\tau > 0$), $(R_1)^3 \sim \tau^{1+\alpha_1}$, $(R_2)^3 \sim$

$\sim \tau^{1+\alpha_2}$, $(R_3)^3 \sim \tau^{1+\alpha_3}$. For positive τ ($\lambda > 0$), according to (21) and (13), $h_1 > 0$ and $h_2 > 0$, so that as τ increases R_1 and R_2 increase monotonically; the quantity h_3, however, changes sign at $\lambda = \lambda^*$ ($\operatorname{ch}2\lambda^* = -\alpha_3$), so that R_3 first decreases and then increases.

In the completely anisotropic solution we have, for $\lambda > 0$, $h_3 < h_1 < h_2$.

In contrast to the case of complete anisotropy, under axial symmetry two types of solutions are possible, corresponding to the values $\alpha_1 = 1$ and $\alpha_1 = -1$. In the solution with $\alpha_1 = 1$ the character of the variation of R_1, R_2, and R_3 is analogous to the preceding one; for $\tau > 0$ ($\lambda > 0$) here $h_1 = h_2 > h_3$. In the solution with $\alpha_1 = -1$ ($\alpha_2 = 2$, $\alpha_3 = -1$) the nature of the singularity at $\tau = 0$ is such that $R_1 = \text{const}, R_3 \neq 0, R_2 = 0$; for $\tau > 0$ ($\lambda > 0$) here $0 < h_1 = h_3 < h_2$, so that as τ increases, R_1, R_2, and R_3 increase [4].*

As $|\tau| \to \infty$ the solution (20) becomes isotropized: $R_1 \sim R_2 \sim R_3 \sim |\tau|^{1/2}$.

4. **The equation of state** $e = p$**.** From (7) in this case we have

$$(-g)^{1/2} = \text{const}\cdot\tau. \tag{22}$$

The origin of the time count has been chosen so that $g = 0$ at $\tau = 0$. From (5) we obtain

$$(R_1)^3 = \text{const}\, \tau^{1+\alpha_1}, \quad (R_2)^3 = \text{const}\, \tau^{1+\alpha_2}, \quad (R_3)^3 = \text{const}\, \tau^{1+\alpha_3}, \tag{23}$$

where the constants α_1, α_2, and α_3 are connected in this case by one relation following from (22):

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \tag{24}$$

From (4) we obtain the expression for e:

$$e = c^2 \left[6 - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)\right]/144\pi k \tau^2. \tag{25}$$

We shall consider only solutions with nonnegative values of $e = p$. Then, according to (25), $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \leq 6$. Let

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 6q^2, \quad q = \text{const}, \quad 0 \leq q \leq 1. \tag{26}$$

Analogously to (13) and (14), in the present case we shall have

$$-q \leq \alpha_1 \leq q, \quad 2q \geq \alpha_2 \geq q, \quad -q \geq \alpha_3 \geq -2q, \tag{27}$$

$$\begin{gathered} \alpha_1 = 2q \sin\gamma, \quad \alpha_2 = 2q\sin(\gamma + 2\pi/3), \quad \alpha_3 = 2q\sin(\gamma + 4\pi/3), \\ -\pi/6 \leq \gamma \leq \pi/6, \quad 0 \leq q \leq 1. \end{gathered} \tag{28}$$

Equality signs correspond to axial symmetry, and strict inequality to complete anisotropy.

The moment $\tau = 0$ corresponds to a singularity ($e = \infty$). At $\tau = 0$, by (27), $R_1 = R_2 = 0$, while R_3 may be equal to ∞, to 0, or to a finite nonzero value, depending on the values of the constants γ and q. For $q = 0$ we obtain the isotropic flat solution. For $q = 1$, according to (26) and (25), $e = p = 0$, and the solution (23) is the anisotropic solution for the vacuum [5]. The solution (23) for the equation of state $e = p$, in contrast to the solutions (10) and (20) for $p = 0$ and for $e = 3p$, does not become isotropized as $|\tau| \to \infty$ (this also applies to the anisotropic solution in the vacuum).

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* For $p = 0$ the axially symmetric solutions (10) with $\alpha_1 = 1$ and with $\alpha_1 = -1$ reduce to one another. In these solutions, however, the singularity occurs at the values τ_1 and τ_2, with different behavior of the solution in their vicinity.

Note: Figure translations are in progress. See original paper for figures.