TRANSFORMATIONS PRESERVING HARMONIC COORDINATES

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Abstract

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TRANSFORMATIONS PRESERVING HARMONIC COORDINATES

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The paper studies transformations that take any harmonic coordinate system in a Riemannian space into a harmonic one. It is proved that if these transformations form a continuous group, then only linear transformations have the property of taking any harmonic coordinate system into a harmonic one. In contrast to ([1]), we consider harmonic coordinates locally, without imposing conditions at infinity. The treatment is carried out by the well-known method of group properties of differential equations ([2]).

Let V_n denote an arbitrary fixed n-dimensional Riemannian space. We shall call a coordinate system $\{x\}$ in V_n harmonic if in these coordinates the components of the metric tensor of the space V_n satisfy the equations

$$\frac{\partial}{\partial x^k}\left(\sqrt{|g|}\, g^{ik}\right) = 0 \qquad (i = 1, \dots, n). \tag{1}$$

In any Riemannian space there exists an infinite set of harmonic coordinates. Denote by Γ the set of all harmonic coordinates of the space V_n and introduce the following

Definition. A transformation

$$x'^i = \varphi^i(x) \qquad (i = 1, \dots, n) \tag{2}$$

is called a **transformation preserving harmonic coordinates** in V_n, if it maps the set Γ into itself.

In this definition it is assumed that by x^i are denoted arbitrary harmonic coordinates, so that all harmonic coordinates are transformed by means of the same functions φ^i $(i = 1, \dots, n)$. In contrast to the definition introduced here, in ([1]) by a transformation preserving harmonic coordinates in the space V_n is meant such a coordinate transformation as takes one fixed harmonic coordinate system into a harmonic one. In the same sense the problem of the uniqueness of harmonic coordinates was studied in ([3]). Below we shall use only the definition introduced here when speaking of transformations preserving harmonic coordinates.

On coordinate transformations (2) that preserve harmonic coordinates in the space V_n, we impose the following restriction. We require that they form a continuous group H, generated by one-parameter continuous local Lie groups.

Denote by $E(x, g)$ the arithmetic space whose point coordinates are x^i $(i = 1, \dots, n)$ and g^{ik} $(i, k = 1, \dots, n)$. In view of the transformation formula for the components of the metric tensor

$$g'^{ik} = g^{mn} \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^k}{\partial x^n} \qquad (i, k = 1, \dots, n), \tag{3}$$

the continuous group H of coordinate transformations (2) defines a continuous

the group of transformations of the space $E(x, g)$ into itself, which we shall denote by $\overline{H}$.

Theorem 1. *Whatever the Riemannian space V_n, the largest continuous group H of transformations (2) preserving harmonic coordinates in the space V_n has order $n(n+1)$ and consists of all linear transformations*

$$x'^i = a^i_k x^k + b^i \qquad (i = 1, \dots, n), \tag{4}$$

where a^i_k, b^i $(i, k = 1, \dots, n)$ are arbitrary constants.

The proof of the theorem is based on the following two lemmas.

Lemma 1. *The group $\overline{H}$ corresponding to the linear transformations (4) is the largest group of continuous transformations of the form (2), (3) admitted by equations (1) in the sense of S. Lie.*

Proof. Introducing the quantities

$$h^{ik} = \sqrt{|g|}\, g^{ik} \qquad (i, k = 1, \dots, n)$$

and denoting

$$\partial h^{ik} \partial x^j \equiv h^{ik}_{,j} \qquad (i, k, j = 1, \dots, n),$$

we write equations (1) in the form

$$h^{ik}_{,k} = 0 \qquad (i = 1, \dots, n). \tag{5}$$

Instead of (3) we shall then have the transformations

$$J\left(\frac{\partial x'}{\partial x}\right) h'^{ik} = h^{mn} \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^k}{\partial x^n} \qquad (i, k = 1, \dots, n), \tag{6}$$

where $J(\partial x' / \partial x)$ is the Jacobian of the coordinate transformation. Write the infinitesimal operator of the group $\overline{H}$ in the form

$$X = \xi^i(x) \frac{\partial}{\partial x^i} + \eta^{ik}(x, h) \frac{\partial}{\partial h^{ik}}. \tag{7}$$

From (6), passing to infinitesimal transformations, we obtain

$$\eta^{ik}(x, h) = h^{il} \frac{\partial \xi^k}{\partial x^l} + h^{kl} \frac{\partial \xi^i}{\partial x^l} - h^{ik} \frac{\partial \xi^l}{\partial x^l} \qquad (i, k = 1, \dots, n). \tag{8}$$

Prolong the operator (7), under condition (8), to the derivatives $h^{ik}_{,j}$ by the formulas (2)

$$\widetilde{X} = X + \zeta^{ik}_j \frac{\partial}{\partial h^{ik}_{,j}},$$

$$\zeta^{ik}_j = \frac{\partial \eta^{ik}}{\partial x^j} + h^{mn}_{,j} \frac{\partial \eta^{ik}}{\partial h^{mn}} - h^{ik}_{,l} \frac{\partial \xi^l}{\partial x^j} \qquad (i, k, j = 1, \dots, n). \tag{9}$$

Consider the invariance conditions for equations (5)

$$\widetilde{X} h^{ik}_{,k} \Big|_{h^{jk}_{,k}=0} = \zeta^{ik}_k \Big|_{h^{jk}_{,k}=0} = 0 \qquad (i = 1, \dots, n).$$

Substituting the values of ζ^{ik}_k from formula (9) and taking equations (5) into account, we obtain that these conditions take the form

$$h^{kj} \left(\partial^2 \xi^i / \partial x^k \partial x^j\right) = 0 \qquad (i = 1, \dots, n). \tag{10}$$

Equations (10) must be satisfied identically in the variables x^i $(i = 1, \dots, n)$ and h^{ik} $(i, k = 1, \dots, n)$. Therefore, taking into account the symmetry of the quantities h^{ik}, we obtain from equations (10)

$$\partial^2 \xi^i / \partial x^k \partial x^j = 0 \qquad (i, k, j = 1, \dots, n).$$

This means that the quantities ξ^i $(i = 1, \dots, n)$ are linear functions of x^k $(k = 1, \dots, n)$, which proves the lemma.

Lemma 2. *In order that the group H of transformations (2) map the set Γ into itself, it is necessary and sufficient that the group $\overline{H}$ be admitted by equations (1) in the sense of S. Lie.*

Proof. a) Sufficiency. If the group $\overline{H}$ is admitted by equations (1), then from the general theory ([2]) it follows that (2), (3) transform any solution of equations (1) into some solution of (1). In particular, the solution realizing the space V_n under consideration is also transformed into a solution of equations (1) by the group $\overline{H}$, and the transformations of the group $\overline{H}$ preserve V_n. This means precisely that the group H transforms Γ into itself.

b) Necessity. It is obvious that linear transformations of coordinates do not take us out of the set Γ. Therefore, by the theorem on the equivalence of quadratic forms of the same signature, at any point (x_0) of the space V_n the components of the metric tensor may be assigned arbitrary values without leaving the set Γ. This means that, by ranging over the various elements of the set Γ, we can obtain any point of the space $E(x, g)$.

Let us now consider some continuous group G of transformations of the space $E(y)$ of variables $y^1, \dots, y^N$ into itself, generated by one-parameter continuous Lie groups G_1. Let the manifold $\mathfrak{M} \subset E(y)$ be given by the equations

$$\psi^\sigma(y) = 0 \qquad (\sigma = 1, \dots, s),$$

and let $\Phi \subset \mathfrak{M}$ be a submanifold. If every transformation $T \in G$ takes any point $y \in \Phi$ into some point $Ty \in \mathfrak{M}$, then for all operators X of the group G the equations

$$X\psi^\sigma(y)\big|_{y\in\Phi} = 0 \qquad (\sigma = 1, \dots, s). \tag{11}$$

hold. Indeed, by assumption, any operator X is generated by some one-parameter continuous local Lie group G_1 with transformation parameter a (chosen so that the value $a = 0$ corresponds to the identity transformation). Therefore we have

$$\psi^\sigma(T_a y) = 0 \quad \text{for all } y \in \Phi \text{ and } T_a \in G_1.$$

Hence, by the well-known formula ([2]),

$$X\psi^\sigma(y) = \frac{\partial}{\partial a}\psi^\sigma(T_a y)\big|_{a=0} \qquad (\sigma = 1, \dots, s),$$

we obtain (11).

Let us apply this result to the case where, as the group G, we take the group obtained by prolonging $\overline{H}$ to the derivatives ${h^i}_{,j}{}^k$, and, instead of $E(y)$, $\mathfrak{M}$, and Φ, respectively, the prolonged space $E(x, h, h^i{}_{,j}{}^k)$, the manifold defined by equations (5), and the submanifold of this manifold corresponding to the set Γ. Writing equations (11), we obtain

$$\widetilde{X} h^i{}_{,k}{}^k \big|_{h^j{}_{,k}{}^k=0} = 0 \qquad (i = 1, \dots, n) \tag{12}$$

for $\{x\} \in \Gamma$.

In view of the property of Γ noted above, to pass through any point of the space $E(x, g)$, or, what is the same, of the space $E(x, h)$, equations (12) must be satisfied identically in the variables x^i $(i = 1, \dots, n)$ and h^{ik} $(i, k = 1, \dots, n)$. But only these variables enter into equations (12). Therefore from (12) there will follow also equations (10), which are the necessary and sufficient condition for the invariance of equations (5). Lemma 2, and hence also Theorem 1, is proved.

Above we restricted ourselves to consideration of groups of transformations of the form (2), (3). However, one may pose the problem of finding the broadest group G of continuous transformations of the form

$$x'^i = \varphi^i(x, g),$$

$$g'^{ik} = \psi^{ik}(x, g) \qquad (i, k = 1, \dots, n), \tag{13}$$

formed by one-parameter continuous local Lie groups, which is admitted by equations (1). Since equations (1) reduce to linear homogeneous equations (5), the group G admitted by these equations contains arbitrary functions. Namely, G contains the subgroup G_0 of transformations

$$g'^{ik} = a \left| \det \left\| \sqrt{|g|}\, g^{mn} + \varphi^{mn} \right\| \right|^{1/(2-n)} \left(\sqrt{|g|}\, g^{ik} + \varphi^{ik}(x) \right) \quad (i, k = 1, \dots, n),$$

where $\varphi^{ik}(x)$ is an arbitrary solution of equations (5), and a is a constant. G_0 is a normal divisor of the group G admitted by equations (1); therefore one may consider the factor group G/G_0.

Theorem 2. *The factor group G/G_0 of the widest continuous group G admitted by equations (1) has order $n(n+2)$ and consists of the transformations*

$$x'^i = \frac{a^i_k x^k + b^i}{c_k x^k + 1},$$

$$g'^{ik} = \left[J\left(\frac{\partial x'}{\partial x} \right) \right]^{4(n+1)(n-2)} g^{mn} \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^k}{\partial x^n} \quad (i, k = 1, \dots, n), \tag{14}$$

where a_k^i, b^i, c_k $(i,k=1,\dots,n)$ are arbitrary constants.

In order to show that the transformations (14) are indeed admitted by equations (1), it is convenient to use the notation of these equations in the form (5). In the case of the transformations (14), instead of expressions (8) for the operator (7) we have

$$\eta^{ik} = h^{il}\frac{\partial \xi^k}{\partial x^l} + h^{kl}\frac{\partial \xi^i}{\partial x^l} - \frac{n+3}{n+1}h^{ik}\frac{\partial \xi^l}{\partial x^l} \quad (i,k=1,\dots,n).$$

Substituting them into (9) and writing the invariance conditions for equations (5), we obtain the equations

$$\frac{\partial^2 \xi^i}{\partial x^k \partial x^j} - \frac{1}{n+1}\delta_k^i \frac{\partial^2 \xi^l}{\partial x^j \partial x^l} - \frac{1}{n+1}\delta_j^i \frac{\partial^2 \xi^l}{\partial x^k \partial x^l} = 0 \quad (i,k,j=1,\dots,n),$$

with general solution

$$\xi^i = -c_k x^k x^i + a_k^i x^k + b^i \quad (i=1,\dots,n),$$

in accordance with (14).

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1. V. A. Fock, *The Theory of Space, Time, and Gravitation*, Moscow, 1961.

2. L. V. Ovsyannikov, *Group Properties of Differential Equations*, Publishing House of the Siberian Branch of the Academy of Sciences of the USSR, 1962.

3. F. I. Frankl, *Uspekhi Mat. Nauk*, **11**, no. 3 (1956).

Note: Figure translations are in progress. See original paper for figures.