INTEGER POINTS IN PERTURBED CIRCLES

MATHEMATICS

1968

SovietRxiv

Abstract

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UDC 511.5

MATHEMATICS

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INTEGER POINTS IN PERTURBED CIRCLES

(Presented by Academician Yu. V. Linnik on July 4, 1967)

1. A generalization of the classical problem on the number of integer points in a circle is the problem on the number of integer points in an arbitrary plane convex domain. As early as 1906, W. Sierpiński ([1]) (by the method of G. F. Voronoi) obtained, for the number of integer points $A(R)$ in a circle of radius R, the estimate $|A(R)-\pi R^2| \ll R^{2/3}$; subsequently this result was strengthened by a number of authors and replaced by an inequality of the form $|A(R)-\pi R^2| \ll R^{\vartheta}$ with $\vartheta < 2/3$. For arbitrary convex domains, Sierpiński' s estimate was generalized by I. M. Vinogradov, Van der Corput, and K. C. Herz ([2]). For the number of integer points $A(B)$ in a convex domain B with sufficiently smooth boundary $\widetilde{B}$, the inequality

$$|A(B) - V(B)| \ll \rho^{2/3}(\widetilde{B}), \tag{1}$$

holds, where $V(B)$ is the area of the domain B, and $\rho(\widetilde{B})$ is the maximal radius of curvature of the boundary $\widetilde{B}$ of the domain B. The smoothness conditions on $\widetilde{B}$ are of essential importance for the formulation of the problem: M. V. Jarník showed (([3]), p. 303) that among domains with boundary $\widetilde{B} \subset C^2$ there exist some for which estimate (1) cannot be improved. Yu. V. Linnik suggested that, by imposing stronger conditions on the smoothness of the boundary of the domain, one can obtain a power improvement of the general estimate (1); more specifically, he posed the problem of strengthening inequality (1) at least for some class of domains. In the present note a positive answer to this question is given for domains differing little from a circle. Namely, the following is true.

Theorem 1. *Let the boundary of the domain B in polar coordinates (ρ, φ) be defined by the equation*

$$\rho = R + r(R, \varphi), \tag{2}$$

where the function $r(R, \varphi)$ satisfies the conditions

$$\frac{\partial^k}{\partial \varphi^k} r(R, \varphi) = o(R), \qquad 0 \le k \le 4 \tag{3}$$

(the relations (3) hold uniformly in φ). Let $A(B)$ be the number of integer points in B, and $V(B)$ the area of the domain B. Then as $R\to\infty$

$$|A(B)-V(B)| \ll R^{112/169}\ln^{37/18}R.$$

2. We outline the main stages of the proof. Denote by $F(B,t)$ the Fourier coefficient of the characteristic function of the domain B with index t (t is a two-dimensional vector with integer components n and m), i.e.

$$F(B,t)=\iint\limits_{B} e^{2\pi i(nu+mv)}\,du\,dv,$$

and put $H(B)=A(B)-V(B)$. Applying Parseval' s equality to the convolution of the characteristic functions of the domains $B+D$ and $-D$, where $D=\delta B$, δ is sufficiently small, one easily obtains the inequality

$$H(B)\ll V(B)\delta+\sum_{t\neq 0}{}' F(B+D,t)F^{*}(D,t)V^{-1}(D)=$$

$$=V(B)\delta+\sum_{t\neq 0}\Phi(B,t,\delta). \tag{4}$$

Next, using the asymptotic estimates of the Fourier coefficients of the characteristic function $F(B,t)$ of the convex domain B (for large $|t|$) from [4], we reduce the problem of estimating $H(B)$ to the problem of estimating sums of the form

$$S_{a,b}=\sum_{a<n\leqslant 2a}\sum_{b<m\leqslant 2b} e^{2\pi iP(n,m)}\psi(n,m), \tag{5}$$

where

$$P(n,m)=\sup_{z\in B}(t\cdot z)=\sup_{(u,v)\in B}(nu+mv),$$

$$\psi(n,m)=F^{*}(D,t)V^{-1}(D)/|t|^{3/2}K^{1/2}(\theta);$$

$K(\theta)$ denotes the curvature of the curve (2) at the point at which $\sup(t\cdot z)$ is attained. We estimate trivially the sum of the terms of the series (4) corresponding to indices t for which at least one coordinate is $\geqslant T$. The contribution of these terms turns out to be

$$\ll \delta^{-3/2}(RT)^{-1}. \tag{6}$$

Using Abel's transformation and estimating the derivatives $\psi(n, m)$ with respect to n and m, we obtain the lemma

Lemma 1. For $S_{a,b}$ the inequality

$$S_{a,b} \ll \left[\frac{\delta^2 R^{5/2}}{(a^2+b^2)^{-1/4}} + \frac{\delta R^{3/2}}{(a^2+b^2)^{1/4}} + \frac{R^{1/2}}{(a^2+b^2)^{3/4}} \right] \max \sum_{a<n\leqslant a'} \sum_{b<m\leqslant b'} e^{2\pi i P(n,m)},$$

where $a' \leqslant 2a,\ b' \leqslant 2b$.

To estimate the resulting trigonometric sum of the form

$$S = \sum_{a<n\leqslant a'} \sum_{b<m\leqslant b'} e^{2\pi i P(n,m)} \tag{7}$$

for $|n| < T,\ |m| < T$, we use the method applied by E. Titchmarsh in [5, 6]. Namely, we shall approximate our sum (7) by a sum of double integrals and estimate the arising integrals.

3. Let $C = \max(a, b)$. Taking $T > R^{55/169}$, we remove from the square L ($|n| < T,\ |m| < T$) the smaller square M ($|n| < R^{55/169},\ |m| < R^{55/169}$), and for the sum (5) extended over M we shall use the trivial estimate

$$S_M \ll R^{112/169}. \tag{8}$$

For the case $C \geqslant R^{55/169}$ we consider two subcases:

(a) at least one of the inequalities

$$a' - a < R^{54/169}, \qquad b' - b < R^{54/169}$$

holds, or

(b)

$$a' - a \geqslant R^{54/169}, \qquad b' - b \geqslant R^{54/169}.$$

For case (a) we trivially obtain

$$S \ll CR^{54/169}.$$

Consider case (b). Applying Lemmas β' and β from [5] to the sum S, we reduce the problem of estimating S to estimating the sum

$$S_1 = \sum_n \sum_m e^{2\pi i q(n,m)},$$

where the function $q(n,m)$ is constructed from $P(n,m)$ in exactly the same way as the function $g(n,m)$ in [6] from the function $\sqrt{n^2+m^2}$. In view of the fact that

$$P(n,m) = \rho^2(\theta)\sqrt{n^2+m^2} \Big/ \sqrt{\rho^2(\theta) + \rho {o'}^2(\theta)} =$$

$$= R(1+o(1))\sqrt{n^2+m^2} = R\sqrt{n^2+m^2} + o(R)\sqrt{n^2+m^2}.$$

estimates obtained in (6) for g_{uu}, g_{uv}, and g_{vv} are valid, respectively, also for q_{uu}, q_{uv}, and q_{vv}. Consequently, one can find an h, depending on C and R, such that if the summation domain S_1 is divided into squares of side h, then for each such square there will exist integers x and y satisfying the conditions:

$$|\mu_u| \le \frac{3}{4}, \qquad |\mu_v| \le \frac{3}{4}, \qquad \mu(u,v) = q(u,v) - xu - yv.$$

Applying Lemma γ from (5) to S_1, extended to an arbitrary such square of side h, we obtain

$$S_1^{(h)} = \sum\sum e^{2\pi i q(n,m)} = \sum\sum e^{2\pi i \mu(n,m)} = \iint e^{2\pi i \mu(u,v)}\,du\,dv + O(h).$$

Estimating the integrals $\iint e^{2\pi i \mu(u,v)}$ and summing over all the squares into which the summation domain S_1 is divided, we finally obtain

$$S \ll C^{16/9} R^{1/18} \ln^{1/18} R.$$

Using Lemma 1, we establish that the sums (5) to which S with condition (a) correspond give a total contribution to

$$\sum_{L-M} \Phi(B,t,\delta)$$

of

$$\delta^2 R^{953/338} T^{3/2} \ln^2 R + \delta R^{615/338} T^{1/2} \ln^2 R + R^{11/169} \ln 2R. \tag{9}$$

The sums (5) to which S with condition (b) correspond contribute in all to

$$\sum_{L-M} \Phi(B, t, \delta)$$

the amount

$$\delta^2 R^{23/9} T^{41/18} \ln^{37/18} R + \delta R^{14/9} T^{23/18} \ln^{37/18} R + R^{5/9} T^{5/18} \ln^{37/18} R. \quad (10)$$

As a result, for $H(B)$, collecting the estimates (6), (8), (9), and (10) and noting that $V(B) \ll R^2$, we establish

$$\begin{aligned} H(B) \ll \delta R^2 + \delta^{-3/2}(RT)^{-1} + \delta^2 R^{23/9} T^{41/18} \ln^{37/18} R + R^{112/169} \\ + \delta^2 R^{953/338} T^{3/2} \ln^2 R + \delta R^{615/338} T^{1/2} \ln^2 R \\ + \delta R^{14/9} T^{23/18} \ln^{37/18} R + R^{5/9} T^{5/18} \ln^{37/18} R. \end{aligned}$$

Choosing $T = R^{59/169}$, $\delta = R^{-226/169}$, we finally obtain:

$$H(B) \ll R^{112/169} \ln^{37/18} R.$$

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Received
27 VI 1967

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