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Abstract

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ON THE LIOUVILLE THEOREM FOR GENERALIZED ANALYTIC FUNCTIONS

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The Liouville theorem known in the theory of analytic functions asserts that any regular solution of the Cauchy–Riemann system of equations, bounded in the whole plane, is a constant. This theorem was extended by I. N. Vekua ([1]) to generalized analytic functions of the class $\mathfrak{A}_{p,2}(E)$, $p > 2$, i.e., to solutions of elliptic systems of the form

$$\partial w/\partial \bar{z} + aw + b\bar{w} = 0, \tag{1}$$

where $a,\ b \in L_{p,2}(E)$, $p > 2$. (Here $z = x + iy$, $w = u + iv$, $\partial/\partial \bar{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$, and the bar denotes complex conjugation.)

For systems not belonging to the class $\mathfrak{A}_{p,2}(E)$, with $p > 2$, this theorem ceases to be true. For example, the system $\partial w/\partial \bar{z} + w = 0$ does not belong to the aforementioned class and has a solution bounded in the whole plane, $w = Ce^{iy}$, different from a constant. An analogous deviation is also observed in the case where one seeks solutions growing at infinity no faster than $|z|^N$.

In the present paper an extension of the Liouville theorem will be given for solutions of elliptic systems with constant coefficients. Namely, the following problem will be investigated: to find solutions, regular in the whole plane, of system (1) satisfying the growth condition

$$|w| \leq C|z|^N, \tag{2}$$

where N is some natural number.

We note that any elliptic system of first order in two unknown functions with constant coefficients can always be reduced to the form (1).

Let us formulate the main result.

Theorem. If $|a|^2 - |b|^2 < 0$, then problem (1)–(2) has only the solution identically equal to zero. If $|a| = |b|$ and $a \neq 0$, it has $N + 1$ linearly independent solutions. If $|a|^2 - |b|^2 > 0$, or if $a = b = 0$, it has $2(N+1)$ linearly independent solutions. Linear independence is understood over the field of real numbers.

Proof. By transformations of the form $w = w_1 e^{i\varphi}$, $z = z_1 e^{i\psi}$, one can always ensure that the coefficients a and b of system (1) are real and positive; moreover, the conditions in the statement of the theorem do not change, since under these transformations the moduli of the coefficients do not change.

Let now w be a solution of our problem; then it can be regarded as a generalized function of the class S' ([2], [3]), whose Fourier transform satisfies the equation

$$\left(\frac{i}{2}\xi + a\right)\tilde{w}(\xi) + b\overline{\tilde{w}(-\xi)} = 0, \qquad \xi = \xi + i\eta. \tag{3}$$

We assume that on basic functions the Fourier transform and its inverse have the form

$$\widetilde{\varphi}(\xi,\eta) = \iint e^{i(x\xi+y\eta)}\varphi(x,y)\,dx\,dy, \qquad \varphi(x,y) = \frac{1}{(2\pi)^2}\iint e^{-i(x\xi+y\eta)}\widetilde{\varphi}(\xi,\eta)\,d\xi\,d\eta.$$

Equation (3) is equivalent to the system of four equations

$$\begin{gathered}
({}^1/{}_2 i\xi + a)\widetilde{w}(\xi) + b\overline{\widetilde{w}}(-\xi) = 0,\\
(-{}^1/{}_2 i\xi + a)\widetilde{w}(-\xi) + b\overline{\widetilde{w}}(\xi) = 0,\\
\bar{b}\,\widetilde{w}(-\xi) + (-{}^1/{}_2 i\bar{\xi} + \bar{a})\overline{\widetilde{w}}(\xi) = 0,\\
\bar{b}\,\widetilde{w}(\xi) + ({}^1/{}_2 i\bar{\xi} + \bar{a})\overline{\widetilde{w}}(-\xi) = 0.
\end{gathered} \tag{4}$$

The determinant of this system is equal to

$$\Delta = \left|-{}^1/{}_4|\xi|^2 + |a|^2 - |b|^2 - {}^1/{}_2 ia(\xi + \bar{\xi})\right|^2. \tag{5}$$

From this the result of the first case follows immediately, since when $|a|^2 - |b|^2 < 0$ the determinant of the system is everywhere nonzero, and consequently $\widetilde{w}$ and w are identically equal to zero.

Let us now consider the second case, when $|a|^2 - |b|^2 = 0$, and $a \neq 0$. Here the determinant vanishes only at the point $\xi = 0$; therefore the support of the generalized function $\widetilde{w}$ consists of the single point $\xi = 0$, and, according to the theorem on the structure of generalized functions with support at one point ([2,4]), $\widetilde{w}$ is a finite linear combination of the δ-function and its derivatives. Consequently, w is a polynomial of degree N in $\bar{z}$ and z. To find the form of this polynomial, represent it as a sum of homogeneous terms

$$w = \sum_{n=0}^{N} P_n, \qquad P_n = \sum_{k=0}^{n} c_{k,n-k} \bar{z}^k z^{n-k} \tag{6}$$

and make a similarity transformation of the independent variable z, $z = \rho z_1$, which will bring our system to the form

$$\partial w / \partial \bar{z} + w + \bar{w} = 0. \tag{7}$$

For P_n we shall have the equations

$$P_N + \bar{P}_N = 0, \qquad P_n + \bar{P}_n = -\partial P_{n+1} / \partial \bar{z}, \qquad n \neq N. \tag{8}$$

Add to system (8) the compatibility equations

$$\operatorname{Im} \partial P_n / \partial \bar{z} = 0, \qquad n = 1, \dots, N, \tag{9}$$

which are necessary and sufficient for its solvability. Hence for P_N we have two equations

$$P_N + \bar{P}_N = 0, \qquad \operatorname{Im} \partial P_N / \partial \bar{z} = 0, \tag{10}$$

whose general solution in our case is the polynomial

$$P_N = p_N (\bar{z} - z)^N,$$

where p_N is any complex number satisfying the equation

$$p_N + (-1)^N \bar{p}_N = 0. \tag{11}$$

Next we apply mathematical induction, going down from N to 0. Suppose P_{n+1} has been found; then P_n satisfies the equations

$$P_n + \bar{P}_n = -\partial P_{n+1} / \partial \bar{z}, \qquad \operatorname{Im} \partial P_n / \partial \bar{z} = 0. \tag{12}$$

Hence

$$P_n = -{}^1/_2 \partial P_{n+1} / \partial \bar{z} + i Q_n, \tag{13}$$

$$Q_n = i \int_0^x \operatorname{Im} \partial^2 P_{n+1} / \partial \bar{z}^2 \, dx + p_n (\bar{z} - z)^n,$$

where p_n is a complex number satisfying the equation

$$p_n + (-1)^{n+1}\overline{p_n} = 0.$$

Thus, at each stage of the calculation a new independent real parameter appears by virtue of (16). Consequently, the general solution will depend on $N+1$ real parameters.

Let us now consider the case $|a|^2 - |b|^2 > 0$. Here the support of the solution of system (4) consists of two points $\pm i \cdot 2c$, $c = \sqrt{a^2 - b^2}$. Consequently,

$$\widetilde{w}(\xi) = \sum_{\alpha,\beta=0}^{N} \left(c_{\alpha\beta} \frac{\partial^{\alpha+\beta}}{\partial\xi^\alpha \partial\eta^\beta} \delta(\zeta - 2ic) + d_{\alpha\beta} \frac{\partial^{\alpha+\beta}}{\partial\xi^\alpha \partial\eta^\beta} \delta(\zeta + 2ic) \right),$$

$$w = Pe^{-i2cy} + Qe^{i2cy},$$

where P and Q are certain polynomials of degree N in $\overline{z}$ and z, satisfying the system of differential equations

$$\partial P/\partial\overline{z} + (c+a)P + b\overline{Q} = 0,$$

$$\partial Q/\partial\overline{z} + b\overline{P} - (c-a)Q = 0. \tag{14}$$

Again we represent P and Q as sums of homogeneous polynomials

$$P = \sum P_n, \qquad Q = \sum Q_n$$

and consider the system obtained for them

$$(c+a)P_N + b\overline{Q}_N = 0,$$

$$b\overline{P}_N - (c-a)Q_N = 0; \tag{15}$$

$$(c+a)P_n + b\overline{Q}_n = -\partial P_{n+1}/\partial\overline{z},$$

$$b\overline{P}_n - (c-a)Q_n = -\partial Q_{n+1}/\partial\overline{z}. \tag{16}$$

It is easy to see that for $b = 0$, $P \equiv 0$, and the solution of our problem is

$$w = \sum_{n=0}^{N} q_n z^n e^{i2ay}, \tag{17}$$

where q_n are arbitrary complex numbers.

For $b \neq 0$, the determinant of system (15) is equal to zero, and systems (15), (16) are equivalent to the system

$$(c+a)P_N + b\overline{Q}_N = 0,$$

$$(c+a)P_n + b\overline{Q}_n = -\partial P_{n+1}/\partial \bar{z}, \tag{18}$$

$$b\overline{P}_n - (c-a)Q_n = -\partial Q_{n+1}/\partial \bar{z}, \qquad n \neq N.$$

For solutions of the latter to exist, it is necessary and sufficient that the compatibility conditions be fulfilled:

$$\frac{\partial P_n}{\partial \bar{z}} - \frac{c+a}{b}\frac{\partial \overline{Q}_n}{\partial z} = 0 \qquad (n = 1, \dots, N). \tag{19}$$

We shall solve system (18) successively, going downward from N to 0. For P_N, from (18), (19) one obtains the equation

$$\frac{\partial P_N}{\partial \bar{z}} + \left(\frac{c+a}{b}\right)^2 \frac{\partial P_N}{\partial z} = 0. \tag{20}$$

To integrate equation (20), we apply the linear transformation of the independent variable

$$z = \left(\frac{c+a}{b}\right)^2 z_1 + \bar{z}_1 \quad \left(z_1 = \left(\frac{b}{c+a}\right)^2 z - \left(\frac{b}{c+a}\right)^4 \bar{z}\right), \tag{21}$$

which, since $c + a \neq b$, is one-to-one. After integration we obtain

$$P_N = p_N \left(\bar{z} - \left(\frac{b}{c+a}\right)^2 z\right)^N, \qquad Q_N = -\frac{a+c}{b} P_N,$$

where p_N is an arbitrary complex number.

Now suppose that the polynomials P_{n+1} and Q_{n+1} have been found. Then from (18) it follows that P_n satisfies the equation

$$\frac{\partial P_n}{\partial \bar{z}} + \left(\frac{c+a}{b}\right)^2 \frac{\partial P_n}{\partial z} = \lambda(\bar{z}, z), \tag{22}$$

where

$$\lambda(\bar{z}, z) = -\frac{c+a}{b^2}\frac{\partial^2 P_{n+1}}{\partial \bar{z}\partial z}$$

is a known polynomial.

After the transformation (21), equation (22) becomes

$$\partial P_n/\partial z_1 = \lambda_1(\bar{z}_1, z_1).$$

Hence

$$P_n = p_n \bar{z}_1^n + \int_0^{z_1} \lambda_1(\bar{z}_1, \xi)\, d\xi,$$

where p_n is an arbitrary complex number.

Passing to the old variables, we obtain an expression for $P_n(z)$, and from the second equation of the system (18), $Q_n(z)$. After each step a new complex constant appears; hence it follows that the general solution of our problem in the present case will depend on $2(N+1)$ real constants.

In the case $a = b = 0$, the classical Liouville theorem for analytic functions is obtained in an analogous way.

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Note: Figure translations are in progress. See original paper for figures.