ON ORIENTED QUASICONFORMAL MAPPINGS IN SPACE

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Abstract

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MATHEMATICS

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ON ORIENTED QUASICONFORMAL MAPPINGS IN SPACE

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1. Let E^n be n-dimensional Euclidean space, $|x|$ the length of the vector $x = (x_1, \dots, x_n) \in E^n$, and G a domain in E^n. By $C_0^\infty(G)$ we shall denote the class of infinitely differentiable finite functions with compact supports $\operatorname{supp} u(x) \subset G$, and by $W_n^1(G)$ the class of vector functions $f(x) = [f_1(x), \dots, f_n(x)]$ having first-order partial derivatives, generalized in the sense of S. L. Sobolev, locally summable to the power n. Put

$$\lambda(x,f) = \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial x_j}\right)^2\right]^{1/2}, \qquad I(x,f) = \frac{\partial(f_1,\dots,f_n)}{\partial(x_1,\dots,x_n)}.$$

A continuous vector function $f(x)$ is said to realize a q-quasiconformal mapping of a domain $G \subset E^n$ if $f(x) \in W_n^1(G)$ and there exists a constant q such that almost everywhere in the domain the inequality

$$\lambda^n(x,f) \le n^{n/2} q^{n-1} |I(x,f)| \tag{1}$$

holds.

In the present note only oriented q-quasiconformal mappings are considered, i.e., mappings for which $I(x,f) \ge 0$ almost everywhere in the domain.

2. Below we shall need the following assertion, which we give here without proof.

Lemma 1. If a continuous vector function $f(x) \in W_n^1(G)$, $u(x) \in C_0^\infty(G)$, then

$$\left|\int_G u(x) I(x,f)\, dG\right| \le C_1 \int_G |\nabla u(x)|\, |f_1(x)|\, \lambda^{n-1}(x,f)\, dG, \tag{2}$$

where C_1 is a constant depending only on n.

The following theorem is a simple consequence of Lemma 1.

Theorem 1. Let a vector function $f(x)$ realize an oriented q-quasiconformal mapping of a domain $G \subset E^n$.

Then for any compact set $F \Subset G$ the inequality

$$\|\lambda(x, f)\|_{L^n(F)} \leq C_2 \|f_1(x)\|_{L^n(G)} \tag{3}$$

holds, where C_2 depends only on n, q, and $\rho(F, \partial G)$.

This theorem makes it possible to sharpen somewhat the results of Yu. G. Reshetnyak ([1]) and E. D. Callender ([2]) concerning the equicontinuity of q-quasiconformal mappings.

Theorem 2. Under the conditions of Theorem 1, for any compact set $F \Subset G$, for $x', x'' \in F$, one has

$$|f(x') - f(x'')| \leq C_3 |x' - x''|^\alpha, \tag{4}$$

where C_3 is a constant depending only on n, q, $\rho(F, \partial G)$, and $\|f_1(x)\|_{L^n(G)}$, and $0 < \alpha \leq 1$ depends on q and n.

This assertion follows directly from inequality (3) and Theorem 2 of paper ([1]).

Theorem 3. Let $\{f(x)\}$ be a family of oriented $(1+\varepsilon)$-quasiconformal mappings of the n-dimensional $(n > 2)$ ball $G : |x| < 1$, normalized by the conditions: $f(0) = 0$, $f(1) = 1$, and having uniformly bounded norms $\|f_1(x)\|_{L^n(F)}$ on every compact set $F \subset G$.

Then there exists a universal function $\mu(\varepsilon, r) \geq 0$, defined for all $0 \leq r < 1$ and possessing the properties: $\mu(\varepsilon, r) \to 0$ as $\varepsilon \to 0$, for which

$$|f(x) - x| \leq \mu(\varepsilon, r). \tag{5}$$

The proof of this theorem is carried out analogously to ([3]), using the preceding theorem and Theorem 1 of paper ([4]).

For homeomorphic q-quasiconformal mappings this assertion was obtained by M. A. Lavrent'ev ([5]) (under some additional restrictions) and by Yu. G. Reshetnyak ([3]).

3. Let F be a compact set belonging to a domain $G \subset E^n$. Define its conformal capacity by

$$\operatorname{cap}_G F = \inf \int_G |\nabla u(x)|^n \, dG, \tag{6}$$

where the infimum is taken over all functions $u(x) \in C_0^\infty(G)$ equal to 1 on F.

In proving the following two theorems we use

Lemma 2. If the vector function $f(x)$ realizes an oriented q-quasiconformal mapping of a domain $G \subset E^n$, then for any compact set $F \subset G$ one has

$$\|\lambda(x, f)\|_{L^n(F)} \leq C_4 \sup_G |f_1(x)| (\operatorname{cap}_G F)^{1/n}, \tag{7}$$

where C_4 depends only on q and n.

The following two theorems contain results that overlap with already known earlier (see ([6])) assertions on the impossibility of mapping the whole space onto a part and on the removability of isolated singularities. Although our results are not generalizations of the known ones, they contain new information.

Theorem 4. If the vector function $f(x)$ realizes an oriented q-quasiconformal mapping of the entire space E^n and if

$$\max_{|x|\leq r} |f_1(x)| = o\left(\ln^{(n-1)/n} r\right) \quad (r \to \infty), \tag{8}$$

then

$$f(x) \equiv \text{const}.$$

Theorem 5. Let the vector function $f(x)$ realize an oriented q-quasiconformal mapping of the domain $G : 0 < |x| < 1$, and suppose that

$$\max_{r \leq |x| \leq 1/2} |f_1(x)| = o\left(\ln^{(n-1)/n} r\right) \quad (r \to 0), \tag{9}$$

$$\lim_{x \to 0} |f(x)| \neq \infty. \tag{10}$$

Then $f(x)$ is continuous at the origin.

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Note: Figure translations are in progress. See original paper for figures.