A RAREFACTION WAVE AND THE EXPANSION OF A MEDIUM IN A VARIABLE GRAVITATIONAL FIELD

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Abstract

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A RAREFACTION WAVE AND THE EXPANSION OF A MEDIUM IN A VARIABLE GRAVITATIONAL FIELD

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The study of gas motion in a gravitational field has numerous applications both in astrophysics and in technology ([1,2]). Here we shall consider a new problem, in which a gas cloud ejected from some gravitating body (for example, a star) begins to expand. If the dimensions of this cloud are small in comparison with the dimensions of the body, then it may be assumed that the acceleration of gravity is sufficiently uniform throughout the cloud. In this case the problem can be solved approximately by considering the motion of the center of mass of this cloud relative to the main body and the expansion of the gas relative to this center of mass.

The acceleration of gravity varies according to a quite definite law $a = a(r) = a(t)$, since the trajectory of the center of mass is prescribed.

We shall consider the problem of the motion of a one-dimensional nonstationary ideal gas relative to the center of mass; we direct the acceleration of gravity along the axis $x = r$, then

$$a = a(r) = GM/x^2. \tag{1}$$

Here M is the mass of the gravitating body, G is the gravitational constant, and the coordinate of the center of gravity of the gas mass x_C is related to time in the following way:

$$x_C^3 = A(t + t_0)^2 + x_0^3. \tag{2}$$

Then the equations of gas dynamics take the form ([3,4])

$$u_t + uu_x + \frac{2}{k-1}cc_x = -\frac{GM}{(At^2 + x_0^3)^{2/3}},$$

$$c_t + uc_x + \frac{k-1}{1} cu_x = 0. \tag{3}$$

We determine the arbitrary constant A from the initial condition of the problem. Setting, for $t = 0$, $x = R$, the radius of the gravitating body, from (2) we have

$$x_0^3 = R^3 - At_0^2. \tag{4}$$

(2) and (4) give

$$x = [A(\tau^2 - t_0^2) + R^3]^{1/3}, \tag{5}$$

where $\tau = t + t_0$, whence it is obvious that the expansion velocity of a gas particle is

$$u = {}^2/_3 A\tau / [A(\tau^2 - t_0^2) + R^3]^{2/3}. \tag{6}$$

Specifying, for $t = 0$, $u = u_0$, we find

$$t_0 = 3u_0 R^2 / 2A. \tag{7}$$

The acceleration of the dispersing gas particles can be expressed by the equality:

$$g = \frac{2}{3}\{A[A(\tau^2 - t_0^2) + R^3] - \frac{4}{3}A^2\tau^2\} / [A(\tau^2 - t_0^2) + R^3]^{5/3}.$$

If the initial acceleration of the dispersing particles is known (i.e., if we set $g = g_0$ at $t = 0$), then from the last equality we find

$$t_0^2 = \frac{3R^3(2A - 3R^2 g_0)}{8A^2}. \tag{8}$$

Comparing (8) with (7), we determine the quantity

$$A = \frac{3}{2}R(2u_0^2 + Rg_0). \tag{9}$$

Expression (2) will now take the form

$$x_3 = \frac{3}{2}(2u_0^2 + Rg_0)\left[(t^2 + (2u_0 Rt)/(2u_0^2 + Rg_0))^2 + R^3\right]. \tag{10}$$

Let us return to the solution of the problem. Suppose that previously the gas was in a state of adiabatic equilibrium; then

$$c^2 = c_{\text{n}}^2 + (k-1)ax - (k-1)GM/R, \tag{11}$$

where c_{n} is the initial speed of sound in the section $x = R$; R is the upper boundary of the gas.

Let, at the instant $t = 0$, in the section $x = R$, the expansion of the gas begin; then for the first rarefaction wave we have the special solutions

$$u + \int a\,dt = -\frac{2}{k-1}c, \tag{12}$$

$$x = (u-c)t + t\int a\,dt - \iint a\,dt\,dt. \tag{13}$$

The minus sign before c indicates that the front of the rarefaction wave moves in the direction of decreasing x.

Substituting the value $a(t)$ into (12), we have:

$$u + GM\int \frac{d\tau}{(A\tau^2 + x_0^3)^{2/3}} = -\frac{2}{k-1}c.$$

Putting $A\tau^2/x_0^3 \ll 1$ and restricting ourselves to the first two terms of the expansion in a series, we find that

$$u + \frac{GM}{x_0^2}\int \frac{d\tau}{(1+N\tau^2)} = -\frac{2}{k-1}c.$$

Integrating, we arrive at the expression:

$$u + B_0 \operatorname{arc\,tg}\sqrt{N}\,t = -\frac{2}{k-1}c + \text{const},$$

where

$$B_0 = \frac{GM}{x_0^2}\sqrt{N}, \qquad N = \frac{2}{3}\frac{A}{x_0^3}.$$

Since at $t = 0$, $u = u_0$, $c = c_{\text{n}}$, it follows that

$$\text{const} = \frac{2}{k-1}c_{\text{n}} + u_0 + B_0 \operatorname{arc\,tg}\sqrt{N}t_0,$$

which gives

$$u - u_0 + B_0(\operatorname{arc\,tg}\sqrt{N}t - \operatorname{arc\,tg}\sqrt{N}t_0) = \frac{2}{k-1}(c_{\text{n}} - c). \tag{14}$$

Now one can find an explicit expression for (13)

$$x = (u-c)t + t\int \frac{GM\,d\tau}{x_0^2(1+N\tau^2)} - \iint \frac{GM}{x_0^2}\frac{d\tau\,d\tau}{(1+N\tau^2)},$$

$$x = (u-c)t + B_0 t \operatorname{arctg}\sqrt{N}\tau + C_1 - \int (B_0 \operatorname{arctg}\sqrt{N}\tau + C_1)\,d\tau.$$

$$x = (u-c)t - c_{\text{n}}t + R + B_0 t_0(\operatorname{arctg}\sqrt{N}t_0 - \operatorname{arctg}\sqrt{N}\tau) + \frac{B_0}{2N}\left[\ln(1+N\tau^2) - \ln(1+Nt_0^2)\right]. \tag{15}$$

The rarefaction-wave front moves according to the law determined from the condition $u = u_0$

$$c = \sqrt{c_{\text{n}}^2 + (k-1)ax - (k-1)GM/R};$$

in this case, from (14) we obtain

$$ax = \frac{k-1}{4}B_0(\operatorname{arctg}\sqrt{N}\tau - \operatorname{arctg}\sqrt{N}t_0)^2 -$$

$$-B_0 c_{\text{n}}(\operatorname{arctg}\sqrt{N}\tau - \operatorname{arctg}\sqrt{N}t_0) + GM/R,$$

whence the rarefaction-wave front is expressed by the law

$$x = 4GM/\{(k-1)B_0(\operatorname{arctg}\sqrt{N}\tau - \operatorname{arc}tg\sqrt{N}t_0)^2 -$$

$$-B_0 c_{\text{n}}(\operatorname{arctg}\sqrt{N}\tau - \operatorname{arctg}\sqrt{N}t_0) + GM/R\}. \tag{16}$$

If we consider the simpler case in which, before the outflow, all the parameters determining the state of the gas are constant, then the rarefaction-wave front has a simpler form. Indeed, when at $t = 0$, $c = c_{\text{n}}$, $u = u_0$, (14) gives $\operatorname{arctg}\sqrt{N}\tau - \operatorname{arctg}\sqrt{N}t_0 = 0$, and this brings the rarefaction-wave front to the form

$$x = u_0 t - 2c_{\text{n}}t + R + \frac{B_0}{2N}\left[\ln(1+N\tau^2) - \ln(1+Nt_0^2)\right]. \tag{17}$$

Since at the expansion front $c = 0$, the gas boundary moves according to the law

$$x = u_0 t - \frac{k-3}{k-1} c_{\text{n}} t + R - B_0 \tau (\operatorname{arctg} \sqrt{N}\tau - \operatorname{arctg} \sqrt{N} t_0) +$$

$$+ \frac{B_0}{2N} [\ln(1 + N\tau^2) - \ln(1 + N t_0^2)]. \quad (18)$$

One can determine the maximum rise of the expanding gas particles. Since at the leading front, when $u = 0$, $c = 0$, solution (14) determines

$$\operatorname{arctg} \sqrt{N}\tau - \operatorname{arctg} \sqrt{N} t_0 = \frac{2}{k-1} \frac{c_{\text{n}}}{B_0} + \frac{u_0}{B_0}, \quad (19)$$

then substitution of (19) into (15) determines the maximum rise of a particle

$$x_{\max} = B - c_{\text{n}} t + \frac{B_0}{2N} [\ln(1 + N\tau^2) - \ln(1 + N t_0^2)], \quad (20)$$

where $B = R - \dfrac{2}{k-1} t_0 c_{\text{n}} - u_0 t_0$.

From equality (19) one can determine the time of maximum rise of the leading boundary of the gas. The energy of the gas before the onset of motion is equal in both cases. This will make it possible to compare the distribution of the initial speed of sound along the gas column.

In the second case the energy is

$$E_0 = \rho_0 c_0^2 l / k(k-1), \quad (21)$$

where ρ_0, c_0 are, respectively, the density and speed of sound of the gas over the entire height l.

In the first case the gas energy is

$$E_1 = \int_R^{R+l} \frac{\rho c^2 \, dr}{k(k-1)}. \quad (22)$$

Here

$$c = (k-1) GM/r, \qquad \rho = \rho \left(\frac{c}{c}\right)^{2/(k-1)}. \quad (23)$$

Integrating (22), taking (23) into account, for E_1 we obtain the equality:

$$E_1 = [\rho\, c^2/k(k-1)]\left(1+\frac{l}{R}\right) l. \tag{24}$$

Comparing (21) and (24), we arrive at the result

$$c_{\,0} = c\,\left(1+\frac{k-1}{2k}\frac{l}{R}\right), \tag{25}$$

from which it follows that, for $l/R \ll 1$, the ratio $c_{\,0}/c$ differs little from unity, which makes reliable the proposed method for investigating the flow of a gas in a variable Newtonian gravitational field during the ejection of gas masses.

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Note: Figure translations are in progress. See original paper for figures.