ANALYTIC REPRESENTATIONS OF THE FULL LINEAR GROUP OVER A FINITE FIELD

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ANALYTIC REPRESENTATIONS OF THE FULL LINEAR GROUP OVER A FINITE FIELD

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This note studies one important class of irreducible representations of the group $G_n = GL(n, K_q)$, where K_q is the field of order q.

Let $Z_n \subset G_n$ be the subgroup of upper triangular matrices with ones on the diagonal.

Definition 1. A one-dimensional representation χ of the subgroup Z_n is called **degenerate** if there exists another one-dimensional representation χ' of Z_n for which $\operatorname{Ker}\chi' \underset{\neq}{\subset} \operatorname{Ker}\chi$.

It is not hard to see that any one-dimensional representation χ of the group Z_n has the form

$$\chi(\xi) = \chi_0(a_1\xi_{12} + a_2\xi_{23} + \cdots + a_{n-1}\xi_{n-1,n}),$$

where χ_0 is a nontrivial additive character of the field K_q, $a_i \in K_q$, and the representation χ is nondegenerate if and only if all $a_i \neq 0$.

It is known (^{2}) that the restriction of any irreducible representation T of the group G_n to Z_n contains at least one one-dimensional representation.

Definition 2. A representation T of the group G_n is called an **analytic representation** (briefly, an A-representation) if the restriction of T to Z_n contains no degenerate one-dimensional representations.

Introduce two subgroups of the group G_n: H_n is the subgroup of matrices $\|a_{ij}\|$ for which

$$a_{n1} = a_{n2} = \cdots = a_{n,n-1} = 0, \quad a_{nn} = 1;$$

$B_n \subset H_n$ is the subgroup of matrices $\|a_{ij}\|$ for which

$$a_{11} = a_{22} = \cdots = a_{nn} = 1$$

and all off-diagonal elements, except the elements of the last column, are equal to zero.

Proposition 1. *With the exception of the special case* $q = n = 2$, *the group* H_n *has one irreducible representation* T_0 *of dimension*

$$(q-1)(q^2-1)\cdots(q^{n-1}-1) = k_n(q);$$

the dimensions of the remaining irreducible representations of H_n *are smaller than* $k_n(q)$.

Proof. H_n is the semidirect product of the commutative normal divisor B_n and the subgroup G_{n-1}. Therefore (^{4}) any irreducible representation T of the group H_n is induced by a representation μL of the group $B_n A_\mu$, where μ is a one-dimensional representation of B_n, $A_\mu \subset G_{n-1}$ is the subgroup of those $g \in G_{n-1}$ such that $\mu(gbg^{-1}) = \mu(b)$, $b \in B_n$, and L is an irreducible representation of A_μ. To finish the proof one must apply a simple induction on n.

Proposition 2. *The restriction of an irreducible* A*-representation of the group* G_n *to* H_n *coincides with* T_0.

Proof. We first show that in the restriction of any irreducible representation of the group H_n different from T_0 to Z_n there is a degenerate one-dimensional representation. Suppose this has already been proved for the group H_{n-1}.

Any irreducible representation T of the group H_n is induced by a representation μL of the group $B_n A_\mu$. If μ is the identity representation, then for any one-dimensional representation χ of the group Z_n contained in the restriction of T to Z_n, $a_{n-1} = 0$, and hence χ is degenerate. Suppose now that μ is a nonidentity representation. Then A_μ is isomorphic to H_{n-1}. If $\dim T < k_n(q)$, then $\dim L < k_{n-1}(q)$, and hence the restriction of L to Z_{n-1} contains a degenerate one-dimensional representation. It follows at once that the restriction of T to Z_n also contains a degenerate one-dimensional representation. Therefore the restriction of any A-representation of the group G_n to H_n is a multiple of T_0.

On the other hand, every nondegenerate one-dimensional representation χ of the group Z_n enters into the restriction of T to Z_n no more than once (^{2}). Therefore the restriction of T to H_n coincides with T_0, and the proposition is proved.

The representation T_0 of the group H_n can be realized in the following way. Fix a nondegenerate character

$$\chi(\xi) = \chi_0(\xi_{12} + \cdots \zeta_{n-1,n})$$

of the group Z_n. The representation T_0 is defined in the space E_χ of functions on G_{n-1} for which

$$f(\zeta g) = \chi(\zeta) f(g), \qquad \zeta \in Z_{n-1}, \quad g \in G^*_{n-1},$$

with scalar product

$$(\varphi_1, \varphi_2)_{n-1} = |G_{n-1}|^{-1} \sum_{g \in G_{n-1}} \varphi_1(g)\overline{\varphi_2(g)}$$

($|G|$ is the number of elements in the finite group G). The operator $T_0(h)$, where $h \in H_n$, $h = ba$, $b \in B_n$, $a \in G_{n-1}$, acts by the formula

$$T_0(h)f(g) = \chi(gbg^{-1})f(ga). \tag{1}$$

This representation is clearly unitary. From (1) it follows that the restriction of T_0 to $G_{n-1} \subset H_n$ coincides with the representation T_χ of the group G_{n-1} induced by the one-dimensional representation χ of the group Z_{n-1}. From (^{2}) follows

Proposition 3. *The restriction of an irreducible A-representation T of the group G_n to G_{n-1} contains each irreducible component no more than once.*

In what follows, an essential role is played by the so-called Bessel functions, introduced in (^{1,2}). They are defined as follows.

Consider the restriction of the regular representation of the group G_n to those functions $f(g)$ on G_n for which

$$f(\zeta g) = \chi(\zeta)f(g), \qquad \zeta \in Z_n, \quad g \in G_n.$$

According to (^{2}), in the space of such functions there exists a unique invariant subspace E_T, the representation in which is equivalent to T. In E_T there exists, up to a factor, a unique vector $J(g)$ for which

$$J(g\zeta) = J(g)\chi(\zeta), \qquad \zeta \in Z_n.$$

This vector is called the Bessel function. From the definition it follows easily that

1. $$J(\zeta_1 g \zeta_2) = \chi(\zeta_1\zeta_2)J(g).$$
2. $$\overline{J(g)} = J(g^{-1}).$$
3. $$J(e) = 1.$$
4. $$J * J = cJ; \quad * \text{ is convolution on the group } G_n,$$
$$f_1 * f_2(g) = |G_n|^{-1} \sum_{g' \in G_n} f_1(g')f_2(gg'^{-1}).$$

In what follows we normalize J so that $c = k_n(g)$. It is easy to establish the following expression for $J(g)$ in terms of the character ψ of the representation T:

$$J(g) = |Z_n|^{-1} \sum_{\zeta \in Z_n} \chi(\zeta)\psi(g\zeta^{-1}).$$

Let us now determine where the function $J(g)$ is concentrated. It is known (^{3,5}) that any element $g \in G_n$ can be written in the form

$$g = \zeta_1 \delta s \zeta_2,$$

where $\zeta_1, \zeta_2 \in Z_n$, $\delta \in D_n$ is a diagonal matrix, and $s \in W_n$ is a permutation matrix. Hence, in view of property 1, it suffices to determine for which δs one has $J(\delta s) \neq 0$. Denote by s_k the matrix of order k in which the second diagonal consists of ones, and all other entries are zero. By

$$s_{i_1,\dots,i_l}(\delta_1, \dots, \delta_l)$$

we denote the matrix

$$s_{i_1,\dots,i_l}(\delta_1, \dots, \delta_l) = \begin{pmatrix} \delta_1 s_{i_1} & & 0 \\ & & \ddots & \\ 0 & & \delta_l s_{i_l} \end{pmatrix} s_n,$$

where $\delta_i \in K_q^*$, $i_1, \dots, i_l$ are integers, and

$$i_1 + \dots + i_l = n.$$

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* We assume that G_{n-1} is embedded in G_n so that if $g \in G_{n-1}$, $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \in G_n$.

Proposition 4. If $J(\delta s) \neq 0$, then $\delta s = s_{i_1,\dots,i_l}(\delta_1, \dots, \delta_l)$ for some i_k and δ_k.

The proof follows easily from property 1.

Let φ_i be an orthonormal basis in the space E_χ, and let $k_{ij}(h)$ be the corresponding matrix elements of the representation T_0 of the group H_n. Let $f_i(g)$ be the image of φ_i in E_T under the isometric mapping $\tau : E_\chi \to E_T$, intertwining the operators of the representation T_0.

Lemma. For any i and j the equality

$$|H_n|^{-1} \sum_{h \in H_n} J(gh) \overline{k_{ij}(h)} = \gamma_i f_j(g), \qquad (2)$$

holds, where γ_j are the coefficients in the expansion

$$J(g) = \sum_i \gamma_i f_i(g).$$

Proof.

$$J(gh) = \sum_i \gamma_i f_i(gh) = \sum_{i,j} \gamma_i k_{ij}(h) f_j(g),$$

$$\sum_{h\in H_n} J(gh)\overline{k_{i_3j_3}(h)} = \sum_{i,j}\gamma_i f_j(g)\sum_{h\in H_n} k_{ij}(h)\overline{k_{i_3j_3}(h)} = |H_n|\gamma_i f_j(g).$$

Theorem. The matrix elements $K_{ij}(g)$ of the representation T in the basis $\{f_i\}$ are given by the formula

$$K_{ij}(g) = c\sum_{a_1,a_2\in G_{n-1}} J(a_2ga_1^{-1})\varphi_i(a_1)\overline{\varphi_j(a_2)},$$

where $c = |G_{n-1}|^{-3}|Z_{n-1}|^{-1}$.

Proof. Transform formula (2):

$$\gamma_i f_j(g) = |H_n|^{-1}\sum_{h\in H_n} J(gh)\overline{k_{ij}(h)} =$$

$$= |H_n|^{-1}|G_{n-1}|^{-1}\sum_{\substack{a,a_0\in G_{n-1}\\ b\in B_n}} J(gba_0)\overline{\chi(aba^{-1})}\varphi_i(aa_0)\overline{\varphi_j(a)} =$$

$$= |H_n|^{-1}|G_{n-1}|^{-1}\sum_{\substack{a,a_0\in G_{n-1}\\ b\in B_n}} J(ga_0)\overline{\varphi_i(aa_0)}\varphi_j(a)\chi(a_0^{-1}ba_0)\overline{\chi(aba^{-1})}.$$

Make the substitution $aa_0 = a'$, $aba^{-1} = b_1$. Then

$$\gamma_i f_j(g) = |G_{n-1}|^{-1}|H_n|^{-1}\sum_{a,a'\in G_{n-1}} J(ga^{-1}a')\overline{\varphi_i(a')}\varphi_j(a)\sum_{b_1\in B_n}\overline{\chi(b_1)}\chi(a'^{-1}b_1a').$$

If $\varphi^0\in E_\chi$ is such a function that $\varphi^0(a) = 0$ for $a\in Z_{n-1}$, $\varphi^0(e) = 1$, and β_i are its coordinates in the basis φ_i, then

$$\sum_{a\in G_{n-1}} J(ga^{-1})\varphi_j(a) = c_0 f_j(g), \qquad c_0 = |G_{n-1}|^2\sum_i\gamma_i\overline{\beta_i}. \tag{3}$$

The matrix elements $K_{ij}(g)$ of the representation T are computed by the formula

$$K_{ij}(g_0) = |G_n|^{-1}\sum_{g\in G_n} f_i(gg_0)\overline{f_j(g)}.$$

Substituting (3), we obtain

$$K_{ij}(g_0) = |G_n|^{-1}c_0^{-2}\sum_{\substack{g\in G_n\\ a_1,a_2\in G_{n-1}}} J(gg_0a_1^{-1})\overline{J(ga_2^{-1})}\varphi_i(a_1)\overline{\varphi_j(a_2)} =$$

$$= c_0^{-2}\mathop{{\sum}'}_{a_1,a_2\in G_{n-1}}\varphi_i(a_1)\overline{\varphi_j(a_2)}\,|G_n|^{-1}\mathop{{\sum}'}_{g\in G_n} J(ga_2g_0a_1^{-1})\overline{J(g)}.$$

In view of properties 2 and 4 of the Bessel function, the inner sum is equal to

$$|G_n|^{-1} \sideset{}{'}\sum_{g\in G_n} J(ga_2g_0a_1^{-1})\overline{J(g)} = J * J(a_2g_0a_1^{-1}) = \bar{k}_n(q)J(a_2g_0a_1^{-1}).$$

Finally we obtain that

$$K_{ij}(g_0) = \bar{k}_n(q)c_0^{-2} \sideset{}{'}\sum_{a_1,a_2\in G_{n-1}} J(a_2g_0a_1^{-1})\varphi_i(a_1)\overline{\varphi_j(a_2)}.$$

To find c_0, observe that $\sum_i \overline{\gamma_i}\bar{\beta}_i = (\tau^{-1}J, \varphi^0)_{n-1}$. But $\tau^{-1}J$, as is not difficult to see, is a multiple of φ^0, $|\varphi^0| = \sqrt{|\bar{Z}_{n-1}|\,|G_{n-1}|^{-1}}$ and

$$|J| = \left(|G_n|^{-1} \times \sum_g J(g)\overline{J(g)}\right)^{1/2} = \sqrt{\bar{k}_n(q)}\sqrt{J(e)} = \sqrt{\bar{k}_n(q)}.$$

Therefore

$$c_0 = \sqrt{|\bar{Z}_{n-1}|\bar{k}_n(q)|G_{n-1}|^3},$$

and the theorem is proved.

The theorem just proved shows the importance of the Bessel function for the study of representations of the group G_n. We shall now give an expression for the Bessel function of an irreducible A-representation of the group G_3.*

Consider the extension K_{q^3}/K_q. Each irreducible A-representation of the group G_3 is determined by a multiplicative character π of the field K_{q^3}, not identically equal to one on K_q. For $\sigma \in K_{q^3}$ introduce the symmetric functions

$$P_1(\sigma) = \sigma + \sigma^q + \sigma^{q^2}; \quad P_2(\sigma) = \sigma^{q+1} + \sigma^{q^2+1} + \sigma^{q^2+q}; \quad P_3(\sigma) = \sigma^{q^2+q+1}.$$

It is clear that for any $\sigma \in K_{q^3}$, $P_i(\sigma) \in K_q$ $(i = 1, 2, 3)$, and the equation

$$Q_\sigma(\lambda) = \lambda^3 - P_1(\sigma)\lambda^2 + P_2(\sigma)\lambda - P_3(\sigma) = 0$$

has roots $\sigma, \sigma^q, \sigma^{q^2}$.

Using the formulas for the characters of irreducible representations of the group G_3, given in ([6]) (see also ([7])), one can obtain the following expressions for the function $J(g)$:

$$J(s_3(\lambda)) = \pi(\lambda),$$

$$J(s_{1,2}(\lambda_1, \lambda_2)) = q^{-2} \sum_{\sigma^{q^2+q+1} = \lambda_1 \lambda_2^2} \chi_0(\lambda_2^{-1} P_1(\sigma)) \pi(\sigma),$$

$$J(s_{2,1}(\lambda_1, \lambda_2)) = q^{-2} \sum_{\sigma^{q^2+q+1} = \lambda_1^2 \lambda_2} \chi_0(-\lambda_1^{-1} \lambda_2^{-1} P_2(\sigma)) \pi(\sigma),$$

$$J(s_{1,1,1}(\lambda_1, \lambda_2, \lambda_3)) = q^{-3} \sum_{\sigma^{q^2+q+1} = -\lambda_1 \lambda_2 \lambda_3} u(\sigma) \pi(\sigma) - q^{-2} \pi(\lambda_2) \delta(\lambda_1 \lambda_3 + \lambda_2^2),$$

where

$$u(\sigma) = \sum_{\substack{\zeta_1, \zeta_2 \in K_q \\ \zeta_1 \zeta_2 = -\lambda_3^{-1} \lambda_2^{-2} Q_\sigma(\lambda_2)}} \chi_0(\zeta_1 + \zeta_2),$$

$\delta(x)$ is the Kronecker symbol.

According to Proposition 4, $J(\delta s) = 0$ for the remaining δs. In view of property 1, the formulas written down make it possible to find $J(g)$ for all $g \in G_3$.

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1. I. M. Gelfand, M. I. Graev, DAN, **146**, No. 4 (1962).
2. I. M. Gelfand, M. I. Graev, DAN, **147**, No. 3 (1962).
3. I. M. Gelfand, M. A. Naimark, Tr. Mat. Inst. im. V. A. Steklova, AN SSSR, **36** (1950).
4. G. Mackey, Sborn. per. Matematika, **6**, No. 6 (1962).
5. K. Chevalley, Sborn. per. Matematika, **2**, No. 1 (1958).

6. R. Steinberg, Canad. J. Math., **3**, No. 2, 225 (1951).

7. J. A. Green, Trans. Am. Math. Soc., **80**, 402 (1955).

* In (1) the Bessel function was computed for the group G_2.

Note: Figure translations are in progress. See original paper for figures.

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