ON THE THEORY OF SEMISYNTOPOGENIC SPACES

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Abstract

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ON THE THEORY OF SEMISYNTOPOGENIC SPACES

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For the definitions see ([1]).

§ 1. A **semisyntopogenic space** is a pair $[E; S]$, where E is some set, and S is a nonempty set of semitopogenic orders on E satisfying two axioms:

(S_1) for any two orders $<_1, <_2 \in S$ there is a stronger order $<_3 \in S$.

(S_2) if $< \in S$, then there exists an order $<_1 \in S$ such that $<_1^2 \supset <$.

If every order $< \in S$ is topogenic, then the space $[E; S]$ is called **syntopogenic**. Such spaces were studied by A. Császár ([1]). It turns out that the concept of a semisyntopogenic space is essentially broader than the concept of a syntopogenic space (for the corresponding example, though in another connection, see § 3).

Having a semisyntopogenic space $[E; S]$, one may speak of S-open sets ($A \subset E$ is S-open if $A < A$ for some $< \in S$), S-closed sets ($A \subset E$ is S-closed if $S[A] \subset A$; here $S[A]$ denotes the set of limit points of the set A with respect to S: $(x \in S[A]) \iff (\forall B)(B \in V(x) \Rightarrow B \cap A \neq \emptyset)$; we assume here that the set $V(x)$ of neighborhoods of the point x with respect to the structure S forms a filter).

Having obtained the notions of an S-open and an S-closed set, one may define, for example, the notion of connectedness of a semisyntopogenic space $[E; S]$. It turns out that a generalization of the classical concept of a connected topological space is possible in two nonequivalent ways; namely, the definition of a connected space by means of S-open sets is strictly included in the definition of a connected space by means of S-closed sets. It is interesting to note the following

Proposition. *If a semisyntopogenic space $[E; S]$, in which the S-open and S-closed sets are mutually complementary, is S-disconnected, then there exists an $(S; S')$-continuous mapping of this space onto a discrete space $[E'; S']$.*

A corresponding sufficient condition could not be formulated so simply.

§ 2. Let X be some set, and let $[E; S]$ be a semisyntopogenic space; by $F(X; E)$ denote the set of mappings from X into E. Take some order $<$ and define the relation $<'$ between subsets of the set $F(X;E)$ as follows: $A <'' B$ means that the set A consists of such and only such mappings $u : X \to E$ for which $u(X) \subset A_1$, and the set B consists of such and only such mappings $v : X \to E$ for which $v(X) \subset B_1$, with $A_1 < B_1$,

$$(A_2 <' B_2) \iff (\exists A_1)(\exists B_1)(A_2 \subset A_1 <'' B_1 \subset B_2).$$

It turns out that $<'$ is a semitopogenic order, and the set

$$S' = \{<' : < \in S\}$$

is a semisyntopogenic structure on $F(X;E)$. Let

$$\mathfrak{X} = (X_\alpha)_{\alpha \in A}$$

be a nonempty family of nonempty subsets of the set X; by φ_α denote the mapping $u \to u|X_\alpha$ (from $F(X;E)$ to $F(X;E_\alpha)$); struct-

the structure $S'_{\mathfrak{X}}$ on $F(X;E)$ is defined as the supremum of the family of structures $\varphi_\alpha^{-1}(S'_\alpha)$, $\alpha \in A$. It is interesting to note the construction of the structure $S'_{\mathfrak{X}}$.

Let

$$\varphi : \ u \to (u|X_\alpha)_{\alpha \in A}$$

be the canonical mapping of the set $F(X;E)$ into the set

$$\prod_{\alpha \in A} F(X_\alpha; E);$$

it turns out that the structure $S'_{\mathfrak{X}}$ is majorized by the structure

$$\varphi^{-1}\left(\prod_{\alpha \in A} S'_\alpha\right),$$

and if $S'_{\mathfrak{X}}$ is syntopogenous, then these two structures are isomorphic.

Theorem 1. Let X be some set, $[Y;S]$ a semisyntopogenous space, and F a subset of the set $F(X;Y)$ consisting of constant mappings. The spaces $[Y;S]$ and $[F;S'_{\mathfrak{X}}|F]$ are isomorphic.

Theorem 2. If the space $[F(X;Y);S'_{\mathfrak{X}}]$ is T_2-separated, then the space $[Y;S]$ is also T_2-separated.

Theorem 3. Suppose that a nonempty family $\mathfrak{X} = (X_\alpha)_{\alpha \in A}$ of nonempty parts of a set X contains, together with each set X_α, all subsets of this set. If the family $\mathfrak{X} = (X_\alpha)_{\alpha \in A}$ covers the set X, and the semisyntopogenous space $[Y;S]$ is T_2-separated, then the space $[F(X;Y);S'_{\mathfrak{X}}]$ is T_2-separated.

Theorem 4. Let X and Y be some nonempty sets, $[Y_\mu; S_\mu]$, $\mu \in M$, a nonempty family of semisyntopogenous spaces, $(X_\lambda)_{\lambda \in L}$ a family of sets distinct from the empty set, and, for each index $\lambda \in L$,

$$\mathfrak{X}_\lambda = (X_{\alpha_\lambda})_{\alpha_\lambda \in A_\lambda}$$

a nonempty family of nonempty parts of the set X_λ. Let, further,

$$\psi_\lambda : X_\lambda \to X, \qquad f_\mu : Y \to Y_\mu$$

for $\lambda \in L$ and $\mu \in M$. If the set Y is endowed with the structure

$$S = \bigvee_{\mu \in M} f_\mu^{-1}(S_\mu)$$

and one puts

$$\mathfrak{X} = (X_{\alpha_\lambda}), \qquad \alpha^\lambda \in A_\lambda,$$

for all $\lambda \in L$, then the mapping

$$g_{\lambda\mu} : \; u \to f_\mu \circ u \circ \psi_\lambda$$

(where $u \in (X; Y)$) is $(S'_{\mathfrak{X}}; (S_\mu)'_{\mathfrak{X}_\lambda})$-continuous for all $(\lambda; \mu) \in L \times M$.

§ 3. The study of semisyntopogenous structures together with algebraic ones defined on one and the same set and naturally coordinated with one another is of known interest.

We call a semisyntopogenous space $[E; S]$ a **semisyntopogenous group** if E is a group and the mapping

$$(x; y) \to x \cdot y^{-1}$$

is $((S \times S)^p; S)$-continuous.

We shall note only the following fact. From a given structure S on E one can define structures S_π and S_λ, which in the case when $[E; S]$ is a topological group coincide respectively with the right and left uniform structures on E.

Theorem 5. If the group E is commutative, then $[E; S_\pi]$ is a semisyntopogenous group.

Analogously one can define the notions of semisyntopogenous fields and vector spaces. Here are some theorems concerning the latter notion.

Theorem 6. For $(S_1; S_2)$-continuity of a linear mapping f from a semisyntopogenous vector space $[E_1; S_1]$ into a semisyntopogenous vector space $[E_2; S_2]$, it is sufficient that the following conditions be fulfilled: a) the mapping f is $(S'_1; S'_2)$-continuous at zero; b) the family of mappings $x \to x + x_0$, $x, x_0 \in E_i$, $i = 1, 2$, is $(S_i; S_i)$-equicontinuous; c) the structure S_1 is perfect.

Theorem 7. A linear $(S_1; S_2)$-continuous mapping f from a semisyntopogenous vector space $[E_1; S_1]$ into a semisyntopogenous vector space $[E_2; S_2]$ is $((S_1)_\pi; (S_2)_\pi)$-continuous.

Theorem 8. Let $[E; S]$ be an n-dimensional semisyntopogenous vector space over a g-complete* syntopogenous field $[K; S']$. If every S-closed hypersubspace H of the space $[E; S]$ has a semi-

* That is, such that the space $[K; S'_\pi]$ is complete.

syntopogenous completion, and every one-dimensional subspace is isomorphic to the space $[K_s; S']$ (here K_s is the field K, regarded as a vector space over itself), then the space $[E; S]$ is isomorphic to the space

$$\left[\prod_{i=1}^{n} K_s^i;\ \prod_{i=1}^{n} S^i\right], \quad \text{where } K_s^i = K_s,\ S^i = S',\ 1 \le i \le n.$$

Theorem 9. Suppose that: a) $[X; S_2]$ is a barrelled δ-space*; b) $[Y; S]$ is a locally convex R_3-regular syntopogenous space; c) the semisyntopogenous space $[L(X;Y); S'_{\mathfrak{x}} \mid L(X;Y)]$ is a semisyntopogenous vector space, where $L(X;Y)$ is the set of linear $(S_2; S)$-continuous mappings $X \to Y$; moreover we assume that $\mathfrak{x} = (X_\alpha)_{\alpha \in A}$ is the family of all possible finite parts of the set X; d) the set $H \subset L(X;Y)$ is bounded with respect to the structure $S'_{\mathfrak{x}} \mid L(X;Y)$. Then the set H has the following property: if $\{0\} < V$ for $< \in S$, then $\{0\} <_1^{-1} u(V)$ for some $<_1 \in S_2$ and for all mappings u from the set H.

Example. On the set E of real numbers define the relation $<_n$ as follows: $A <_n B$ means that $A \subset G \subset B$, where G is a set open in the classical topology of the number line, whose open bounded component intervals do not exceed n in length. It turns out that: 1) $<_n$ is a semitopogenous order on the set E; 2) $S = \{<_n;\ n = 1, 2, 3, ...\}$ is a semisyntopogenous structure on E; 3) $[E; S]$ is a semisyntopogenous, not syntopogenous (and, a fortiori, not topological!) group; 4) $[E; S]$ is a semisyntopogenous vector space distinct from every topological one.

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REFERENCES

1. Á. Császár, *Grundlagen der allgemeinen Topologie*, Budapest, 1963.

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* By a δ-space we mean a space in which the set of neighborhoods of each point forms a filter.