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REMARKS ON COMPACT CLASSES OF ALGEBRAIC SYSTEMS

MATHEMATICS

1968

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Abstract

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UDC 519.40

MATHEMATICS

S. R. KOGALOVSKII

REMARKS ON COMPACT CLASSES OF ALGEBRAIC SYSTEMS

(Presented by Academician P. S. Novikov on 5 IX 1967)

For any signature μ, by V_μ we shall denote the class of all algebraic systems of this signature, and by L_μ the set of all elementary formulas pertaining to μ.

A class $K \subset V_\mu$ is called **compact** if every finitely satisfiable in K system of sentences from L_μ is satisfiable in K.

A class M of all possible algebraic systems of some signature $\nu \supset \mu$, whose μ-projections are systems from K, is called an **enrichment**, or a **ν-enrichment**, of the class K. Moreover, if $\nu \backslash \mu$ consists of individual constants, then M is called an **individual enrichment** of K. The class K is called **strictly compact** if every individual enrichment of it is compact, and **absolutely compact** if every enrichment of it is compact.

1°. In ([1]) it is shown that compactness of the class K is equivalent to elementary axiomatizability of the class $\mathrm{cl}(K)$, consisting of all algebraic systems elementarily equivalent to systems from K. Hence, from the ultraclosedness of elementarily axiomatizable classes and Theorem 2 of ([2]), it follows that compactness of K is equivalent to the fact that $\mathrm{cl}(K) = \mathrm{cl}(\mathrm{Prod}\, K)$, where $\mathrm{Prod}\, K$ is the closure of K with respect to ultraproducts. The latter is equivalent to the fact that the ultraproduct of every family of systems from K, with respect to any ultrafilter, is elementarily equivalent to some system from K.

Let the cardinal number n be not less than the cardinality of the signature of the class K. Then, if K is closed with respect to filtered products over ultrafilters of weight n, then every enrichment of it whose signature has cardinality not exceeding n^* is compact. Hence follows the absolute compactness of ultraclosed classes. Under the generalized continuum hypothesis, a stronger proposition holds: *if a compact class K is closed with respect to ultrapowers, then it is absolutely compact.*

Indeed, let K^* be a ν-enrichment of the class K for some signature ν, and let Σ be a system of sentences from L_ν finitely satisfiable in K^*. Then Σ is true in some ultraproduct $\mathfrak{M}^*$ of systems from K^*. The μ-projection $\mathfrak{M}$ of

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the system $\mathfrak{M}^*$ is elementarily equivalent to some system $\mathfrak{A}$ from K. There exists an ultrafilter $\langle I, D\rangle$ such that $\mathfrak{M}/D \cong \mathfrak{A}/D$ under the assumption of the generalized continuum hypothesis. $\mathfrak{A}/D \in K$, since K is closed with respect to ultrapowers. Consequently, $\mathfrak{M}/D \in K$, whence $\mathfrak{M}^*/D \in K^*$. In $\mathfrak{M}^*/D$, Σ is true. The proposition is proved**.

From these same arguments and Keisler' s results contained in ([3]), it follows that a compact class closed with respect to taking homod—

* This proposition follows almost directly from the arguments proving Theorem 2 of ([2]).

** From similar arguments it also follows that if a compact class is closed, for example, with respect to taking ultrapowers, then it is absolutely compact without the assumption of the generalized continuum hypothesis. Hence follows Theorem 5 of ([6]).

universal ultrapowers, is strictly compact under the assumption of the generalized continuum hypothesis.

Let (K_i) be a centered system of compact classes. Then $(\mathrm{cl}(K_i))$ is a centered system of elementarily axiomatizable classes, and therefore $\bigcap \mathrm{cl}(K_i) \neq \emptyset$. Hence there follows the existence of an algebraic system which in each K_i has an elementarily equivalent representative. Using this remark, one can find a number of conditions sufficient for the intersection of a centered system of compact classes to be nonempty. For example:

Let C be a closure operation possessing the following property:

$(*)$ *For every algebraic system $\mathfrak{M}$ and every centered system (M_i) of subclasses of the class $\mathrm{cl}(\{\mathfrak{M}\})$, one has $\bigcap C(M_i) \neq \emptyset$.*

Then every centered system of compact C-closed classes has a nonempty intersection.

Property $(*)$ is possessed by closures with respect to ultrapowers* (and hence also closures with respect to ultraproducts**), finer closures consisting in the adjunction of universal homogeneous ultrapowers, etc.

2°. In ([7]) it is shown that direct products preserve compactness. In ([4]) the preservation of compactness by reduced products is proved. The idea of the proof of the latter result can be expressed in the form of Lemma 1.

For every class A we shall denote by $\varphi(A)$ the class consisting of all possible families of elements of A. By V we shall denote the class of all possible algebraic systems (of all possible signatures). For every operation F, defined on some class $W \subset \varphi(V)$, with values in V, and for every class $A \subset V$, we shall denote by $F(A)$ the class $\{F(M) : M \in W \cap \varphi(A)\}$, and by $F^*(A)$ the class consisting of those $F(M) \in F(A)$ such that all systems in M coincide. Let μ and ν be some signatures. We shall call the operation F a $\varphi_{\mu\nu}$-operation if $F(V_\mu) \subset V_\nu$,

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and if from $(\mathfrak{M}_i)_{i\in I} \in W \cap \varphi(V_\mu)$ it follows that $(\mathfrak{M}_i)_{i\in I} \in W$ for every family $(\mathfrak{M}_i)_{i\in I} \in \varphi(V_\mu)$.

Lemma 1. *Let F be a $\varphi_{\mu\nu}$-operation, and G a $\varphi_{\mu\xi}$-operation, having one and the same domain of definition and satisfying the conditions:*

1. *F preserves elementary equivalence in the sense of ([8]).*
2. *For every elementarily axiomatizable class $K \subset V_\mu$, the class $F(K)$ is compact.*
3. *For every sentence $\alpha \in L_\xi$ there exists a sentence $\alpha^* \in L_\nu$ such that, for every $M \in W \cap \varphi(V_\mu)$, the truth of α in $G(M)$ is equivalent to the truth of α^* in $F(M)$.*

*Then, for every class $K \subset V_\mu$, compactness of K implies compactness of $G(K)$.***

Indeed, let K be a compact class. Condition 1 implies

$$\mathrm{cl}(F(K)) = \mathrm{cl}(F(\mathrm{cl}(K))).$$

By condition 2, $F(\mathrm{cl}(K))$ is a compact class. Consequently, $F(K)$ is also a compact class. For every finite satisfiable in $G(K)$ system of sentences $\{\alpha_i\} \subset L_\xi$, the system $\{\alpha_i^*\}$ is finitely satisfiable in $F(K)$ by condition 3. But then, in view of the compactness of $F(K)$, $\{\alpha_i^*\}$ is true on some algebraic system $F(M)$ $(M \in \varphi(K))$. Hence, by condition 3, there follows the truth of $\{\alpha_i\}$ on $G(M)$.

Let, for example, G be some arithmetic operation in the sense of ([6]), and F_0 the operation assigning to each family $(\mathfrak{M}_i)_{i\in I} \in \varphi(V_\mu)$ the system

$$F_0((\mathfrak{M}_i)) = \langle I,\ R_{\theta_0}, \dots,\ R_{\theta_\alpha}, \dots\rangle,$$

where R_{θ_α} are unary pred-

* This follows directly from Theorem 2 of ([2]).

** See ([6]), Theorem 3. This result of Omarov is proved without using the generalized continuum hypothesis.

*** From conditions 1 and 3 of the lemma it follows that G preserves elementary equivalence.

propositions mutually one-to-one with the propositions θ_α from L_μ and such that $F_0((\mathfrak{M}_i)) \vDash R_{\theta_\alpha}(j) \leftrightarrow \mathfrak{M}_j \vDash \theta_\alpha$. From the definition of an arithmetic operation in the sense of (6) it follows immediately that F_0 and G satisfy conditions 1 and 3 of the lemma. It is easy to see that if some class $K \subset V_\mu$ is defined by a system of elementary propositions $\{\theta_\alpha\}_{\alpha\in A}$, then $F_0(K)$ is defined by the system $\{(\forall i)(R_{\theta_\alpha}(i)) : \alpha \in A\}$. Consequently, F_0 satisfies condition 2 of the lemma. Hence, and from Lemma 1, the main result of (6) follows directly: arithmetic operations preserve compactness.

From what has just been proved it follows that, for every arithmetic operation G and every compact class K, the class $G^*(K)$ is compact, as is the class N

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consisting of the G- "products" of such families from $\varphi(K)$ in which there are $\geq n$ (or $\leq n$) systems satisfying an arbitrary fixed proposition $\theta \in L_\mu$, where n is some natural number, etc.

Indeed, if K is a compact class, then $F_0^*(K)$ is elementarily axiomatizable. Hence, and from condition 3 of the lemma, it follows that the class $G^*(K)$ is compact. The compactness of the class N is proved similarly.

We shall denote by B the class of all Boolean algebras such that every element contains an atom and any two elements containing the same atoms coincide. If I is the set of all atoms of an algebra $\mathfrak{B} \in B$, then $\mathfrak{B}$ will be called a Boolean algebra over I.

Let I be a nonempty set, and let T be a set of subsets of I forming a Boolean algebra over I. Any algebraic system

$$\mathfrak{S} = \langle T,\ \wedge,\ \cup,\ \cap,\ -,\ \subseteq,\ M_1, \dots, M_j, \dots \rangle, \tag{1}$$

where $\wedge, \dots, \subseteq$ are the usual set-theoretic operations and relations (restricted to the set T), will be called a generalized algebra of subsets of the set I. If $T = S(I)$, then $\mathfrak{S}$ is called the algebra of subsets of I.

In what follows we shall consider generalized algebras of subsets of an arbitrary fixed signature.

For each generalized algebra of subsets (1), by $F(\mathfrak{S})$ we shall denote the $\varphi_{\mu\omega}$-operation assigning to each family $(\mathfrak{M}_i)_{i\in I} \in \varphi(V_\mu)$ the system

$$\langle T,\ \wedge,\ \cup,\ \cap,\ -,\ \subseteq,\ M_1, \dots, M_j, \dots,\ R_{\theta_0}, \dots,\ R_{\theta_\alpha}, \dots \rangle,$$

whose $\langle \wedge, \dots, \subseteq, M_1, \dots, M_j, \dots \rangle$-projection is $\mathfrak{S}$, and whose $\langle R_{\theta_0}, \dots, R_{\theta_\alpha}, \dots \rangle$-projection of the I-reduct is $F_0((\mathfrak{M}_i))$, and such that, for every α and every $a \in T$ that is not an atom, $\neg R_{\theta_\alpha}(a)$ is true.

Let S be some class of generalized algebras of subsets. Then

$$F(S, K) \underset{df}{=} \bigcup_{\mathfrak{S} \in S} F(\mathfrak{S})(K), \qquad F^*(S, K) \underset{df}{=} \bigcup_{\mathfrak{S} \in S} F(\mathfrak{S})^*(K).$$

Suppose that to each $\mathfrak{S} \in S$ there is assigned a $\varphi_{\mu\xi}$-operation $G(\mathfrak{S})$. We shall say that $\{F(\mathfrak{S}), G(\mathfrak{S})\}_{\mathfrak{S}\in S}$ is a coordinated collection of operations if, for every proposition $\alpha \in L_\xi$, there exists a proposition $\alpha^* \in L_\omega$ such that, for every family $M \in \varphi(V_\mu)$ and every $\mathfrak{S} \in S$, the truth of α in $G(\mathfrak{S})(M)$ is equivalent to the truth of α^* in $F(\mathfrak{S})(M)$. Using the compactness criteria contained in § 1, it is not difficult to show that, if K and S are compact classes, then $F^*(S, K)$ is a compact class. With the aid of this proposition and the lemma one proves

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Theorem 1. Let K be a compact class of systems from V_μ, let S be a compact class of generalized algebras of subsets, and let the operations $F(\mathfrak{S})$, $G(\mathfrak{S})$ $(\mathfrak{S} \in S)$ form a coordinated collection. Then

$$G^*(S, K) = \bigcup_{\mathfrak{S} \in S} G(\mathfrak{S})^*(K)$$

is a compact class.

Corollary. If K is a compact class of systems from V_μ, and S is a compact class of algebras of subsets, then the class $P^*(S, K)$ of generalized powers of systems from K over algebras from S is compact*.

* This result was obtained independently and by another method by G. V. Chudnovskii.

In particular, for every compact class $K \subset V_\mu$ and every subset algebra $\mathfrak{S}$ the class $P^*(\{\mathfrak{S}\}, K)$ is compact (see (6), p. 54).

For generalized products an analogous assertion does not hold. Indeed, let K be the class of models whose signature contains the distinguished constant 0 and an infinite set $\{P_\gamma\}_{\gamma\in\Gamma}$ of unary predicates, defined by the system of sentences $\{\neg P_\gamma(0) \to P_\delta(0)\}_{\gamma\neq\delta;\ \gamma,\delta\in\Gamma}$. It is easy to see that the system $\{\neg P_\gamma(0)\}_{\gamma\in\Gamma}$ is finitely satisfiable, but is not satisfiable in the class $P_{(\omega)}(K)$, consisting of countable direct products of models from K.

Let $\mathfrak{S} = \langle S(I), \wedge, \dots, \subseteq, M_1\rangle$ be a subset algebra such that M_1 has cardinality $\geqslant 1$. Then there exists an elementarily axiomatizable class K such that the class $P(S, K)$, where S is the class of all subset algebras of the same signature as $\mathfrak{S}$, is not compact.

Following (8), every subset algebra $\langle S(I), \wedge, \dots, \subseteq\rangle$ will be called a **simplest subset algebra**.

Theorem 2. *Let K be a compact class, and let S_0 be the class of all simplest subset algebras. Then $P(S_0, K)$ is a compact class.*

The validity of this theorem is easily seen from the proof of Theorem 1 from (7), if, instead of the lemma (FV), one uses Proposition 6.1 from (8).

It is obvious that the intersection of a compact class and an elementarily axiomatizable class is compact. Using this observation and Theorem 2, it is easy to prove the compactness of classes consisting of generalized products over algebras from S_0 of families from $\Phi(K)$ (where K is a compact class), possessing certain properties expressible in the language of the narrow predicate calculus.

An obvious generalization of Theorem 2 is the following proposition: let to each subset algebra $\mathfrak{S} \in S_0$ an operation $G(\mathfrak{S})$ be assigned in such a way that the operations $F(\mathfrak{S})$, $G(\mathfrak{S})$ form a coordinated aggregate. Then the compactness of the class $K \subset V_\mu$ entails the compactness of $G(S_0, K)$.

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Moreover, let S be a subclass of the class S_0, consisting of subset algebras of cardinalities no smaller than the cardinality of the signature of the class K. Then the compactness of K entails the compactness of $G(S, K)$.

Ivanovo Textile Institute
named after M. V. Frunze

Received
1 IX 1967

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