ON A NONLINEAR SYSTEM OF EQUATIONS OF MIXED TYPE

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Abstract

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MATHEMATICS

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ON A NONLINEAR SYSTEM OF EQUATIONS OF MIXED TYPE

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Problem T. In the domain D, find a regular solution of the system

$$E(z_i) = y\,\partial^2 z_i/\partial x^2 + \partial^2 z_i/\partial y^2 = f_i(x, y, z_1, \dots, z_n), \tag{1}$$

taking the prescribed values:

$$z_i\big|_{AC} = \psi_i(t), \qquad z_i\big|_{\Gamma} = \varphi_i(S), \qquad i = 1, 2, \dots, n, \tag{2}$$

where the φ_i are continuously differentiable and the ψ_i are twice continuously differentiable functions.

The domain D is bounded by a Jordan curve Γ in the half-plane $y > 0$, with endpoints at the points $A(0,0)$, $B(1,0)$, and by arcs of the characteristics AC and BC of the equation $E(z) = 0$, if $y < 0$. The functions f_i are assumed to be continuous and to have second derivatives in the domain $x, y \in \overline{D}$, $|z_i| \le c$, and moreover $\partial f_i/\partial z > 0$.

We first establish some properties of solutions of the linear system

$$yz_{xx} + z_{yy} - cz = 0, \tag{3}$$

where $z = (z_1, z_2, \dots, z_n)$ is a vector, and $c(x, y)$ is a positive definite $n \times n$ matrix $c = \|c_{ik}\|$, whose components satisfy the conditions

$$(n-1)(c_{ik} + c_{ki}) \le 2\sqrt{c_{ii}c_{kk}}, \qquad i \ne k. \tag{4}$$

Let

$$R(x, y) = \left(\sum_{k=1}^{n} z_k\right)^{1/2};$$

D_1, D_2 are the parts of the domain D in the half-planes respectively $y > 0$, $y < 0$.

Lemma 1. The function $R(x, y)$ cannot attain a positive maximum at an interior point of the domain D_1 (see ([1])).

Supposing that $R(x, y)$ attains a positive maximum at an interior point P of the domain D_1, we obtain

$$E(R(P)) \leq 0. \tag{5}$$

On the other hand, by virtue of (3),

$$E(R) = \frac{1}{R}\left[y(z_x)^2 + (z_y)^2 + zcz - y(R_x)^2 - (R_y)^2\right]. \tag{6}$$

Consequently, at the point P,

$$E(R) > 0. \tag{7}$$

The contradiction between inequalities (5) and (7) proves the lemma.

Lemma 2. If at some point $x = x_0$ of the segment $(0, 1)$ of the axis $y = 0$, $R(x, y)$ assumes its greatest positive value, and if the values of $R(x, y)$ on Γ are less than $R(x_0, 0)$, then

$$\overline{R}(x_0) = \lim_{y\to 0} \partial R(x_0, y)/\partial y < 0 \tag{8}$$

provided that this limit exists.

For degenerating elliptic equations this proposition is given in work ([2]).

Obviously, $\overline{R}(x_0) > 0$ cannot occur. Suppose that $\overline{R}(x_0) = 0$. Let $\mu > 0$, let d be the diameter of the domain D_1, and let $R(x_0, 0) = 1$. Consider the function

$$u = \varepsilon R/(e^{\mu d} - \varepsilon e^{\mu y}). \tag{9}$$

It is easy to see that the function $u(x, y)$ must have a positive maximum at an interior point P of the domain D_1, and, by virtue of (9),

$$E(R(P)) = \frac{1}{\varepsilon}\left[E(u) - \frac{2\varepsilon e^{\mu y}}{e^{\mu d} - \varepsilon e^{\mu y}} u_y - \frac{\varepsilon \mu e^{\mu y}}{e^{\mu d} - \varepsilon e^{\mu y}} u\right] < 0. \tag{10}$$

On the other hand, at the point P, taking (9) into account, we obtain $R_x = 0$, $R_y = -\mu e^{\mu y} u$, and, by virtue of (3) and (4), from (6) we obtain the inequality $E(R(P)) > 0$, contradicting (10), which proves the lemma.

Lemma 3. In the domain D_2 there exists a unique continuously differentiable solution of system (3), assuming on the boundary the prescribed values

$$z_i\big|_{AC} = \psi_i(t), \qquad z_i\big|_{AB} = \tau_i(x). \tag{11}$$

Let $z_i^{(0)}$ be continuous solutions of the equation $E(z_i) = 0$ in the domain $\overline{D}_2$, assuming on the boundary the values (11). Such solutions are known ([2]).

Let

$$u_i = z_i - z_i^{(0)}. \tag{12}$$

Then the functions u_i satisfy the equation

$$E(u_i) - \sum_{k=1}^{n} c_{ik}(u_k + z_k^{(0)}) = 0 \tag{13}$$

and the homogeneous boundary conditions (11). Setting $u_i^{(0)} = 0$, and

$$u_i^{(m)} = \lambda \int_0^{\xi} d\xi' \int_{\xi'}^{\eta} \frac{V(\xi,\eta;\xi',\eta')}{(\eta'-\xi')^{2/3}} \sum_{k=1}^{n} c_{ik}(u_k^{(m-1)} + z_i^{(0)})\, d\eta', \tag{14}$$

we find that

$$\left|u_i^{(m+1)} - u_i^{(m)}\right| \leq 2cM^{m+1}(\eta-\xi)^{1/3}\eta^{1/3+(m+1)/3}\frac{\xi^m}{m!}, \tag{15}$$

where $c = \max\{|z_k^{(0)}|\}$, $M = 6\lambda Nn$, $N = \max|c_{ik}|$ for all $i, k = 1, 2, \dots, n$.

From the estimates (15) follows the uniform convergence of the sequence $u_i^{(m)}$ to the solution of equation (13).

Lemma 4. If the function $R(\xi,\eta)$ is equal to zero on AC and assumes its greatest positive value at some point $(x_0, 0)$ on AB, then there exists in D_2 a neighborhood of this point in which

$$R(\xi,\eta) < R(x_0, 0). \tag{16}$$

From (12) we have

$$z_i = k\int_0^{\xi} \tau_i(t) \frac{(\eta-\xi)^{2/3}\,dt}{(\eta-t)^{5/6}(\xi-t)^{5/6}} + u_i(\xi,\eta), \tag{17}$$

where $u_i = \lim_{m\to\infty} u_i^{(m)}$.

Assuming that z_i on AB attains its greatest positive value at the point $(x_0, 0)$, and taking into account the estimates (15), we obtain

$$z_i \leq \tau_i(x_0)\left\{1 - \left(\frac{\eta-\xi}{\eta}\right)^{2/3}\left[{}^3/2\,k - 2cM\overline{M}\eta^{1/3}\right]\right\}, \tag{18}$$

where $\overline{M} = \max e^{\xi}\eta^{1/3}$.

It follows from inequality (18) that if $\eta < (3k/4cM\overline{M})^{3/4}$, then $z_i(\xi,\eta) < \tau_i(x_0)$. Since $z_i(\xi,\eta)$ in some neighborhood of the point $(x_0, 0)$, by virtue of continuity, as well as $\tau(x_0)$, is positive, we obtain inequality (16) in an obvious way.

Lemma 5. If the functions $z_i(x,y)$ satisfy equation (3) in the domain D, and are equal to zero on the characteristic AC, then the norm $R(x,y)$ on the segment AB cannot assume a greatest positive value.

By virtue of the first lemma, $R(x,y)$ in D_1 cannot have a positive maximum. Assuming that this maximum is attained on AB, by virtue of (8) and (16) we arrive at a contradiction.

Theorem 1. The solution of problem T for system (3) is unique in D.
The proof follows easily from Lemmas 1, 3, and 5.

Theorem 2. If

$$(n-1)\left(\frac{\partial f_i}{\partial z_k} + \frac{\partial f_k}{\partial z_i}\right) \leqslant 2\left(\frac{\partial f_i}{\partial z_i}\frac{\partial f_k}{\partial z_k}\right)^{1/2},$$

then in the domain D there exists a solution of problem T for equation (1).

Supposing that problem T for system (1) has two solutions z_i and w_i, we obtain that the difference $u_i = z_i - w_i$ satisfies the system

$$E(u_i) = \sum_{k=1}^{n} \frac{\partial f_i}{\partial u_k} u_k,$$

which possesses all the properties of system (3) and, consequently, by virtue of the first theorem has the unique zero solution.

We preface the proof of the existence of a solution by two lemmas.

Lemma 6. In the domain D_1, for system (1) there exists a unique twice continuously differentiable solution which assumes the prescribed values on the boundary

$$z_i\big|_\Gamma = \varphi_i(s), \qquad dz_i/dy = \nu_i(x). \tag{19}$$

The uniqueness of the solution follows from Lemmas 1 and 2. Let $z_i^{(0)}$ be the solution of the equation $E(z_i) = 0$, satisfying conditions (19).

Putting

$$v_i = z_i - z_i^{(0)}, \tag{20}$$

we find that v_i satisfy the equation $E(v_i) = f_i(x, y, v_1 + z_1^{(0)}, \dots,$ $\dots, v_n + z_n^{(0)})$ and the homogeneous conditions (19). Replacing this equation by an integral one and solving it by the method of successive approximations, where $v_i^{(0)} = 0$, and

$$v_i^{(m+1)} = \iint_{D_1} f_i\big(\xi, \eta, v_1^{(m)} + z_1^{(0)}, \dots, v_n^{(m)} + z_n^{(0)}\big)\, G(x, y; \xi, \eta)\, d\xi\, d\eta,$$

we obtain that the functions $v_i = \lim\limits_{m\to\infty} v_i^{(m)}$ solve the posed problem.

For v_i the estimates

$$|dv_i(x, 0)/dx| \leqslant c, \qquad c = \text{const}. \tag{21}$$

are valid.

Lemma 7. In the domain D_2 there exists a unique continuously differentiable solution of system (1), which assumes on the boundary the values

$$z_i\big|_{AC} = \psi_i(t), \qquad z_i\big|_{AB} = \nu_i(x) \tag{22}$$

and has the form

$$z_i = u_i + z_i^{(0)}, \qquad u_i = \lim_{m\to\infty} u_i^{(m)}, \tag{23}$$

where

$$z_i^{(0)}(\xi, \eta) = k \int_0^\xi \nu_i(t)(\xi - t)^{-1/6}(\eta - t)^{-1/6}\, dt + \int_0^b \left(\psi_i' + \frac{\psi_i}{6t}\right) V(\xi, \eta; 0, t)\, dt, \tag{24}$$

$$u_i^{(m+1)}(\xi,\eta) = \lambda \int_0^{\xi} d\xi' \int_{\xi}^{\eta} \frac{V(\xi,\eta;\xi',\eta')}{(\eta'-\xi')^{2/3}}\, f_i(\xi,\eta,u_1^{(m)}+z_1^{(0)},\dots,u_n^{(m)}+z_n^{(0)})\, d\eta'.$$

For $u_i(\xi,\eta)$, when $\eta=\xi=x$, we have the estimates

$$|\, du_i(x,x)/dx \,| \le c, \qquad c = \text{const.} \tag{25}$$

Now from equations (20) and (23), along the segment AB, one can form a system of equations with respect to $\tau_i(x)$ and $\nu_i(x)$.

Eliminating $\tau_i(x)$ from this system, we reduce the solution of problem T to the solution of a singular integral equation with respect to $\nu_i(x)$ (see (2)).

By virtue of the estimates (21) and (25), this singular equation is reduced in the usual way to a Fredholm equation, whose solvability follows from the uniqueness of the solution proved above.

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[2] K. I. Babenko, *On the Theory of Equations of Mixed Type*, Dissertation, Moscow, 1951.

Note: Figure translations are in progress. See original paper for figures.