ON THE EXISTENCE OF COMPLETE HOMEOMORPHISMS FOR SOME QUATERNIONIC EQUATIONS

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 517.946

MATHEMATICS

R. Kh. KRISTALINSKII

ON THE EXISTENCE OF COMPLETE HOMEOMORPHISMS FOR SOME QUATERNIONIC EQUATIONS

(Presented by Academician I. N. Vekua on 25 XI 1966)

To each point (x_0, x_1, x_2, x_3) of four-dimensional Euclidean space E_4 one can associate a certain quaternion $z = x_0 + ix_1 + jx_2 + kx_3$. Let $f(z)$ be a function defined in the space E_4, whose values belong to the algebra of quaternions. Such a function will henceforth be called a **quaternionic function**. To each quaternionic function there corresponds a certain mapping of the space E_4 into itself. If the quaternionic function $w = f(z)$ satisfies (for every z in E_4) some differential equation and the corresponding mapping of the space E_4 onto itself is homeomorphic, then we shall call the function $f(z)$ a **complete homeomorphism** for the equation L.

Questions concerning the existence of a homeomorphism and the scheme for constructing homeomorphisms for elliptic systems with two variables were developed in detail in the monograph (^{1}). V. I. Shevchenko transferred this scheme to the spatial case in the paper (^{2}) and established, in particular, the existence of complete homeomorphisms for one class of quaternionic equations.

We propose another method for solving problems of this kind. It is based on the following result, due to J. Hadamard (^{3}):

Every continuous mapping of Euclidean space into itself which is locally homeomorphic everywhere maps this space homeomorphically onto itself, if every sequence of points tending to infinity is mapped into a sequence of the same kind.

An illustration of the proposed method is given, for example, by the proof of the following theorem:

Theorem 1. Let the quaternionic function $a(z)$ satisfy the following conditions:

1) $a(z)$ vanishes outside some ball T with center at the origin;

2) $L_p(a, T) \leq M_1, \quad p > 4;$

3) $\overline{\lim}_{h\to 0} L_p\left(\frac{a(z+h)-a(z)}{|h|}, E_4\right) \leq M_2$.

Then, if the constants M_1 and M_2 are sufficiently small, for the equation

$$DU = aU, \tag{1}$$

where D is the operator of the form

$$D = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}, \tag{2}$$

there exists a complete homeomorphism.

For the proof, consider in the space $L_p(\widetilde{T})$, where $\widetilde{T}$ is the ball of diameter twice as large as the diameter of the ball T, and with center at the origin, the integral equation:

$$u(z) = -\frac{1}{2\pi^2} \iiiint_{\widetilde{T}} \frac{\overline{t-z}}{|t-z|^4}\, a(t)u(t)\, dV + x_0 + ix_1 + jx_2 - kx_3. \tag{3}$$

If the constants M_1 and M_2 are sufficiently small, then equation (3) will have a unique solution $u(z)$, and the function

$$u_0(z) = -\frac{1}{2\pi^2} \iiint\limits_{T} \frac{\overline{t-z}}{|t-z|^4}\, a(t)u(t)\, dV \tag{4}$$

will belong to the class $C_\alpha(E_4)$, $\alpha = (p-4)/p$. It is obvious that

$$\lim_{|z|\to\infty} u_0(z) = 0, \tag{5}$$

and the function

$$U = u_0(z) + x_c + ix_1 + jx_2 - kx_3 \tag{6}$$

satisfies equation (1) at every point of the space E_4.

Using the fact that the function $u(z)$ is a solution of equation (3), for sufficiently small $|h|$ we obtain the relation

$$\max_{\tau\in T} |u(\tau+h) - u(\tau)| \leq |h| + M \max_{\tau\in T} |u(\tau+h) - u(\tau)|\, L_p(a,T) +$$
$$+ MC(u,T)L_p(|a(\tau+h) - a(\tau)|, E_4), \tag{7}$$

where M is a constant depending on p and on the diameter of the ball T; $C(u,T)$ is the maximum of the modulus of the function $u(z)$ in the ball T.

With the aid of this relation we obtain from (6):

$$\begin{gathered}|U(z+h)-U(z)| \geq \\ \geq |h| - \frac{ML_p(a,T)}{1-ML_p(a,T)}\left[|h| + MC(u,T)L_p(|a(\tau+h)-a(\tau)|,E_4)\right] - \\ - MC(u,T)L_p(|a(\tau+h)-a(\tau)|,E_4). \qquad (8)\end{gathered}$$

Hence it easily follows that, if the constants M_1 and M_2 are sufficiently small, then

$$\lim_{|h|\to 0} \frac{|U(z+h)-U(z)|}{|h|} \geq k > 0 \qquad (9)$$

for any point z of E_4.

Consequently, the mapping of the space E_4 onto itself corresponding to the function $U(z)$ is locally homeomorphic. From Hadamard' s theorem and relation (5) we obtain that $U(z)$ is a complete homeomorphism for equation (1).

Remark. It can be shown that the class of functions satisfying conditions 1) –3) of Theorem 1 coincides with the set of functions from the Sobolev space $W_p'(E_4)$ that vanish outside the ball T.

Applying the above Hadamard theorem, one can also obtain the following result:

Theorem 2. Let the quaternion function $a(z)$ satisfy the following conditions:

1) $L_{p,5}C(a,E_4) \leq N_1$;
2) $$\sup_{\substack{z\in E_4 \\ |t-z|\leq 1}} \frac{|a(z)-a(t)|(1+|z|)}{|z-t|} \leq N_2.$$

If the constants N_1 and N_2 are sufficiently small, then for the equation

$$DU = aU$$

there exists a complete homeomorphism.

In an analogous way one can also obtain the result of V. I. Shevchenko mentioned above.

Smolensk Pedagogical Institute
named after K. Marx

Received
30 VII 1966

References

1. I. N. Vekua, *Generalized analytic functions*, Moscow, 1959.

2. V. I. Shevchenko, DAN, 153, No. 2 (1963).

3. J. Hadamard, Bull. Soc. Math. de France, 34 (1906).

Note: Figure translations are in progress. See original paper for figures.