ON BOUNDED SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract

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M. M. BELOVA

ON BOUNDED SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS OF SECOND ORDER

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In the present work we give some results concerning the existence, uniqueness, and behavior as $x \to \infty$ of solutions, bounded for all x, of systems of the form

$$y'' = F(x, y, y'), \tag{*}$$

where y and $F(x, y, z)$ are n-dimensional vectors. In the case of a single equation, analogous results were obtained in papers ($^{1-3}$); in the case of the system $y'' = \psi(x, y)$, in paper (^{4}). Periodic solutions of system $(*)$ were considered in paper (^{5}). Related questions for first-order systems were studied in the monograph (^{6}).

Theorem 1. Let the equation

$$y'' = f(x, y, y'), \tag{1}$$

be given, where the derivatives $f_y(x, y, z)$, $f_z(x, y, z)$ are bounded in every bounded region of the space x, y, z, and $f_y(x, y, z) > 0$ for all values of z and such values of x, y to which the whole xoy-plane corresponds, except for a set of measure 0.

Then, if equation (1) has a solution bounded for all values of x, it is unique.

The proof is based on the maximum principle, which can be formulated in the form of the following lemma:

Lemma. Let the equation

$$y'' = f(x)y + \varphi(x)y' + \psi(x), \tag{1'}$$

be given, where the functions $f(x)$, $\varphi(x)$, $\psi(x)$ are bounded in some interval (a, b), and almost everywhere in (a, b) the inequalities $f(x) > 0$, $\psi(x) \geq 0$ (≤ 0)

hold. Then a solution of equation $(1')$ cannot have in the interval (a, b) a positive maximum (negative minimum).

Consider the system of equations

$$y'' = A(x)y + (B(x) + c(x)I)y' + F(x), \tag{2}$$

where $y(x)$ and $F(x)$ are n-dimensional column vectors; $A(x)$ and $B(x)$ are square matrices of order n; $c(x)$ is a scalar function; I is the identity matrix. Put

$$|y|_0 = \sup_x |y(x)|, \quad |A| = \sup_{|\xi|=1} |A\xi|,$$

where ξ is an n-dimensional vector,

$$|A|_0 = \sup_x |A(x)|.$$

Theorem 2. Let $|A(x)|$, $|B(x)|$, and $|c(x)|$ be bounded on every finite interval, and

$$((A - \tfrac{1}{4}BB^*)\xi, \xi) > 0 \tag{3}$$

for every vector ξ. Then system (2) has no more than one solution bounded for all x.

The proof is based on applying the maximum principle to the equation for $w = \frac{1}{2}(y - z, y - z)$, where y and z are bounded solutions of (2).

Lemma 1. Suppose

$$(A\xi, \xi) \geq a(\xi, \xi), \quad a = \text{const} > 0, \tag{4}$$

$$|F|_0 < +\infty, \quad \frac{1}{4}|B|_0^2 < a, \tag{5}$$

and the quantity $|A(x)|$ is bounded on every finite interval. Then a solution $y = y(x)$ of system (2), bounded for all values of x, satisfies the inequality

$$|y| \leq \frac{|F|_0}{a - \frac{1}{4}|B|_0^2}. \tag{6}$$

The proof is based on applying the maximum principle to the equation satisfied by the function $w = \frac{1}{2}(y, y)$.

Lemma 2. Suppose the quantities $|A|_0$, $|B|_0$, $|F|_0$, $|c|_0$ are finite. Then for the derivative of a bounded solution $y(x)$ of system (2) the estimate

$$|y'|_0 \le \frac{|A - K^2 I|_0 \, |y|_0 + |F|_0}{K - |B|_0 + |c|_0}, \tag{7}$$

is valid, where K is any number, $K > |B|_0 + |c|_0$.

Lemma 3. Suppose the system is given

$$y'' = A(x)y + c(x)y' + F(x), \tag{8}$$

where $c(x)$ is a scalar function, $|c|_0 < +\infty$, $A(x)$ is a symmetric matrix, and

$$\alpha(\xi, \xi) \le (A(x)\xi, \xi) \le \beta(\xi, \xi), \quad 0 < \alpha \le \beta. \tag{9}$$

Then system (8) has a unique solution, bounded for all values of x, such that

$$|y| \le |F|_0/\alpha. \tag{10}$$

Proof. Uniqueness and inequality (10) follow from Theorem 2 and Lemma 1. Consider a sequence of bounded vector-functions $y_k(x)$ such that $y_0(x) \equiv 0$,

$$y''_{k+1} = \beta y_{k+1} + c(x)y'_{k+1} + F + (A - \beta I)y_k. \tag{11}$$

Such vector-functions exist (3). Applying the maximum principle to the equation for $w_{k+1} = y_{k+1} - y_k$, we obtain

$$|w_{k+1}|_0 \le \frac{\beta - \alpha}{\beta}|w_k|_0, \quad \text{where} \quad \frac{\beta - \alpha}{\beta} < 1.$$

Consequently, $|w_k|_0 \to 0$ as $k \to \infty$. Using Lemma 2 and the equation for w_{k+1}, we prove the convergence of w'_k and w''_k. Hence the assertion of the lemma follows.

Theorem 3. Suppose A is a symmetric matrix satisfying condition (9). Suppose

$$|c|_0 < +\infty, \quad |F|_0 < +\infty, \quad |B|_0 < 2\sqrt{a}. \tag{12}$$

Then there exists a unique solution of system (2), bounded for all values of x, for which inequality (6) and the inequality

$$|y'|_0 \le \frac{4(\sqrt{\beta} + |c|_0 + |B|_0)^2 - |B|_0^2}{4a - |B|_0^2} \, \frac{|F|_0}{\sqrt{\beta}} \tag{13}$$

hold.

Proof. Put $K = \sqrt{\beta} + |B|_0 + |c|_0$ in inequality (7). From inequality (9) and the symmetry of A it follows that

$$|A - K^2 I| \leq (\sqrt{\beta} + |B|_0 + |c|_0)^2 - \alpha.$$

Hence, from (6) and (7), inequality (13) follows.

Consider a sequence of bounded vector functions $y_k(x)$ such that

$$y_0 \equiv 0, \qquad y''_{k+1} = A(x) y_{k+1} + c(x) y'_{k+1} + F(x) + B(x) y'_k. \tag{14}$$

By Lemma 3 such a sequence exists. For $w_{k+1} = y_{k+1} - y_k$, by the maximum principle, using Lemma 2, we shall have

$$|w_{k+1}|_0 \leq \frac{|B|_0 |w'_k|_0}{\alpha}, \qquad |w'_{k+1}|_0 \leq \frac{|B|_0 (\sqrt{\beta} + |c|_0)^2}{\alpha \sqrt{\beta}} |w'_k|_0. \tag{15}$$

Hence, if $|B|_0 < \alpha\sqrt{\beta}/(\sqrt{\beta} + |c|_0)^2$, we obtain $y'_k(x) \to y'(x)$, where $y(x)$ is a bounded solution of system (2).

If $|B|_0 \geq \alpha\sqrt{\beta}/(\sqrt{\beta} + |c|_0)^2$, then the theorem can be proved by considering the approximation

$$y_0(x) = 0,$$

$$y''_{k+1} = (A + LI) y_{k+1} + (B(x) + c(x) I) y'_{k+1} + F(x) - L y_k,$$

where L is such that

$$|B|_0 < \frac{(\alpha + L)\sqrt{\beta} + L}{(\sqrt{\beta} + L + |c|_0)^2}.$$

Theorem 3 generalizes to the system

$$y'' = f(x, y, y') + c(x) y', \tag{16}$$

where $f(x, y, z)$ is a column vector and $c(x)$ is a scalar function.

Denote by f_y the matrix $\|\partial f_i / \partial y_j\|$, and by f_z the matrix $\|\partial f_i / \partial z_j\|$, $i, j = 1, 2, \dots, n$.

Theorem 4. Suppose that $|c|_0 < +\infty$, f_y is a symmetric matrix,

$$\alpha(\xi,\xi) \le (f_y\xi,\xi) \le \beta(\xi,\xi), \qquad |f_z| \le B_0 = \text{const} < 2\sqrt{\alpha}, \qquad |f(x,0,0)|_0 < +\infty.$$

Then system (16) has a unique solution $y(x)$, bounded for all values of x, such that

$$|y(x)|_0 \le \frac{|f(x,0,0)|_0}{\alpha - \frac{1}{4}B_0^2}, \qquad |y'(x)|_0 \le \frac{4(\sqrt{\beta} + B_0 + |c|_0)^2 - B_0^2}{4\alpha - B_0^2}\,\frac{|f(x,0,0)|_0}{\sqrt{\beta}}.$$

Theorem 5. Suppose that the system

$$y'' = f(x,y,y') \tag{17}$$

is given, where

$$(f_y\xi,\xi) \ge \alpha(\xi,\xi), \qquad \alpha = \text{const} > 0, \qquad \frac{1}{4}|f_z|^2 \le \alpha_1 < \alpha,$$

$$\int_0^{+\infty} |f(x,c,0)|^2\,dx < +\infty, \qquad c = (c_1, c_2, \dots, c_m) = \text{const}\,.$$

Then every solution of system (17) bounded on the half-axis $[0,+\infty)$ tends to c as $x \to +\infty$.

Theorem 6. Suppose that the system

$$y'' = f(x,y) + B(x)y', \tag{18}$$

is given, where

$$(f_y\xi,\xi) \ge \alpha(\xi,\xi), \qquad \alpha = \text{const} > 0, \qquad B(x) \text{ is a symmetric matrix},$$

$$\left|\frac{d}{dx}B(x)\right| < 2\alpha, \qquad \int_0^{+\infty} |f(x,c)|^2\,dx < +\infty, \qquad c = \text{const}\,.$$

Then every solution of system (18) bounded on the half-axis $[0,+\infty)$ tends to c as $x \to +\infty$.

Theorem 7. Suppose that for system (16) all the conditions of Theorem 4 are satisfied. Suppose, moreover, that the right-hand side of system (16) is a periodic (almost periodic) function of x for fixed y and z. Then the solution bounded on the whole axis will be periodic (almost periodic).

Moscow Machine-Tool Institute

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Note: Figure translations are in progress. See original paper for figures.

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