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Abstract

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THE CONJUGACY PROBLEM IN THE BRAID GROUP

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1. In 1925 E. Artin ([6]) reduced a number of topological questions to the problems of the identity of words and conjugacy in a certain concrete group, which he called the braid group. He also solved the word problem in the braid group by means of an isomorphic mapping of this group into the automorphism group of a certain free group. In 1945 A. A. Markov ([3]) constructed the theory of braids on a purely algebraic basis, completely excluding geometry from consideration.

An essential question in knot theory and in the theory of closed braids (see ([2]), p. 354) is the conjugacy problem in the braid group. Here partial results were known due to O. Schreier ([7]), who solved the conjugacy problem in the braid group of third order, and A. A. Markov, who solved this problem in the braid group of fourth order (the result was not published).

In the present paper the conjugacy problem is solved in the braid group of arbitrary order.

2. The limits of this article do not allow us to give a detailed description of the proposed algorithm; therefore we shall give formulations of some of the pivotal assertions of this work. We shall adhere to the usual (see, for example, ([1],[4],[5])) definitions and notation used in the study of algorithmic questions of algebra.

We shall define the braid group of $(n+1)$-st order $\mathfrak{B}_{n+1}$ by the generators

$$a_1, a_2, \dots, a_n$$

and defining relations

$$a_{i+1}^{\gamma} a_i^{\delta} a_{i+1}^{\varepsilon} = a_i^{\varepsilon} a_{i+1}^{\delta} a_i^{\gamma},$$

where $i = 1, 2, \dots, n-1$, and for each value of i the parameters $\gamma, \delta, \varepsilon$ run through all values satisfying the conditions

$$|\gamma| = |\delta| = |\varepsilon| = 1, \qquad \gamma = \delta \vee \delta = \varepsilon;$$

$$a_i^\delta a_j^\varepsilon = a_j^\varepsilon a_i^\delta$$

$$(i = 1, 2, \dots, n; \qquad j = 1, 2, \dots, n; \qquad |i - j| > 1; \qquad |\delta| = |\varepsilon| = 1).$$

The following three lemmas describe certain properties of the braid group used in the proof of the main result.

Lemma 1. If $A = 1$ in the group $\mathfrak{B}_{n+1}$, then one can indicate a sequence of elementary transformations of the group $\mathfrak{B}_{n+1}$ without inserting the letter a_n, which transforms the word A into the empty word.

Lemma 2. If $Aa_i^x Ba_j^y = 1$ in the group $\mathfrak{B}_{n+1}$, where $|x| > \partial(AB)$, $|y| > \partial(AB)$, then the numbers x and y have different signs; moreover, if $x > 0$, then $Aa_i^{x-1} Ba_j^{y+1} = 1$ in the group $\mathfrak{B}_{n+1}$, while if $x < 0$, then $Aa_i^{x+1} Ba_j^{y-1} = 1$ in the group $\mathfrak{B}_{n+1}$.

Lemma 3. In the group $\mathfrak{B}_3$ the conjugacy problem is solvable.

3. We shall call a word A minimal in a certain group Γ if every word B equal to the word A in the group Γ has length not less than that of the word A.

Let

$$E \simeq a_n^{\varepsilon_1} L_1 a_n^{\varepsilon_2} L_2 \cdots a_n^{\varepsilon_{s-1}} L_{s-1} a_n^{\varepsilon_s} L_s,$$

where $|\varepsilon_i| = 1$, $s \geqslant 2$, and the words L_i contain no letters a_n. Let

$$L_j = K_{j-1}^{-1} a_{n-1}^{t_j} K_j$$

in the group $\mathfrak{B}_n$, for $j = 1, 2, \dots, s$, $K_0 \simeq K_s$, and the words K_j contain no letters a_{n-1} or a_n and are minimal in $\mathfrak{B}_{n-1}$; moreover,

$$\sum_{j=1}^{s} \partial(L_j) \geqslant \sum_{j=1}^{s} \partial(K_{j-1}^{-1} a_{n-1}^{t_j} K_j).$$

If the word

$$a_n^{\varepsilon_1} a_{n-1}^{t_1} a_n^{\varepsilon_2} a_{n-1}^{t_2} \cdots \cdots a_n^{\varepsilon_{s-1}} a_{n-1}^{t_{s-1}} a_n^{\varepsilon_s} a_{n-1}^{t_s} = 1$$

in the braid group of order three, generated by a_{n-1}, a_n, then we shall say that the word E belongs to the set R.

Lemma 4. *The set R is decidable.*

Each of the subwords $a_n^{\varepsilon_i}$, L_i, $a_n^{\varepsilon_i} L_i$, $L_i a_n^{\varepsilon_i}$, $a_n^{\varepsilon_i} L_i a_n^{\varepsilon_{i+1}}$, and $L_i a_n^{\varepsilon_{i+1}} L_{i+1}$ will be called a component of the word E. If $MN^{-1} \in R$ and the word N^{-1} is a component of the word MN^{-1}, then the relation $M \to N$ will be called a defining relation of the group $\mathfrak{B}_{n+1}$.

Lemma 5. *The set of defining relations of the group $\mathfrak{B}_{n+1}$ is decidable.*

Lemma 6. *If $A = 1$ in the group $\mathfrak{B}_{n+1}$, then one can specify a sequence of defining relations transforming the word A into some word B equal to the identity in the group $\mathfrak{B}_n$.*

4. We shall say that a word X is s-conjugate to a word Y in the group $\mathfrak{B}_{k+1}$, where $k \geqslant 2$, $1 \leqslant s \leqslant k$, if there exists a word Z in the alphabet $\{a_1, a_2, \dots, a_s, a_1^{-1}, a_2^{-1}, \dots, a_s^{-1}\}$ such that $XZ = ZY$ in the group $\mathfrak{B}_{k+1}$. For example, a word X is conjugate to a word Y in the group $\mathfrak{B}_{k+1}$ if and only if the word X is k-conjugate to the word Y in the group $\mathfrak{B}_{k+1}$. We denote the problem of s-conjugacy in the group $\mathfrak{B}_{k+1}$ by $\mathfrak{A}_{k,s}$. We order pairs of natural numbers lexicographically, i.e. set $(k_1, s_1) < (k_2, s_2)$ if and only if one of the following two conditions is satisfied: $k_1 < k_2$, or $k_1 = k_2$ and $s_1 < s_2$.

On the basis of Lemma 6 one proves

Lemma 7. *If $k > 2$ and, for every pair (i, j) preceding the pair (k, s) in the lexicographic ordering, the problem $\mathfrak{A}_{i,j}$ is decidable, then the problem $\mathfrak{A}_{k,s}$ is also decidable.*

From Lemmas 2, 3, and 7 it follows that

Theorem. *There exists an algorithm which, for every natural $n \geqslant 2$ and any two words X and Y in the alphabet $\{a_1, a_2, \dots, a_n, a_1^{-1}, a_2^{-1}, \dots, a_n^{-1}\}$, determines whether the word X is conjugate to the word Y in the group $\mathfrak{B}_{n+1}$.*

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Note: Figure translations are in progress. See original paper for figures.