ON TOPOLOGIES ON PRODUCTS OF GROUPS

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Abstract

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ON TOPOLOGIES ON PRODUCTS OF GROUPS

(Presented by Academician P. S. Aleksandrov on 15 IV 1968)

P. S. Aleksandrov, at a seminar at Moscow University, posed the following problem: do there exist countable topological groups whose spaces are not dyadic (a topological space is called dyadic if it is homeomorphic to a dense subset of some dyadic bicompactum; see ([5])). This problem is directly connected with the question of the dyadicity of the spaces of finally compact topological groups. In this note a natural class of topologies on products of topological spaces is considered, and one necessary and one sufficient condition imposed on topologies from the class under consideration are given in order that the product space be dyadic. In particular, the existence is proved of a countable topological group whose space is not dyadic.

Recall that a point x of a topological space X is called a χ-point if in X there exists a countable bicompact subset in which the point x is nonisolated.

Lemma 1. *If a nondiscrete topological space X is a quotient image of a dyadic space, then every nonisolated point x of the space X is a χ-point in the Čech extension of the space X.*

Proof. Let $f : Y \to X$ be a quotient mapping of a dyadic space Y onto X, existing by the hypothesis of the lemma; let g be its extension to a mapping of the Čech extensions of these spaces; let bY be the dyadic bicompact extension of the space Y, existing by virtue of the dyadicity of Y, and let φ be the canonical mapping of βY onto bY. We must show that a nonisolated point x of X is a χ-point in βX. The set $f^{-1}(x)$ is not open in Y as the inverse image of a nonisolated point under a quotient mapping, and one can choose a point $y \in f^{-1}(x)$ such that

$$y \in [Y \setminus f^{-1}(x)].$$

Since $g|X = f$, y is a limit point of the set $g^{-1}(x)$, and it is not difficult to verify, using the definition of the mapping φ, that $\varphi(y)$ is a limit point of the set $\varphi g^{-1}(x)$. By the Efimov–Katětov theorem ([4]), every nonisolated point of a dyadic bicompactum is a χ-point. One can prove the following strengthening of this assertion: if a point z of a dyadic bicompactum Z is a limit point of a closed subset F, then there exists a subset

$$\{z_n,\ n \in N\} \subset Z \setminus F$$

such that

$$[\{z_n,\ n \in N\}] = \{z_n,\ n \in N\} \cup \{z\},$$

where N denotes the set of natural numbers. Consequently, in our case in the space bY there is a subset

$$\{y_n,\ n \in N\} \subset bY \setminus \varphi g^{-1}(x)$$

such that

$$[\{y_n,\ n \in N\}] = \{y_n,\ n \in N\} \cup \varphi(y).$$

Choose arbitrarily $x_n \in \varphi^{-1}(y_n)$; then

$$\{x_n,\ n \in N\} \subset \beta Y \setminus \varphi^{-1}\varphi g^{-1}(x) \subset \beta Y \setminus g^{-1}(x).$$

Moreover, owing to the continuity and closedness of φ we have

$$\varphi^{-1}\varphi(y) = y \in [\{x_n,\ n \in N\}],$$

and

$$[\{x_n,\ n \in N\}] = \{x_n \in N\} \cup \{y\};$$

$$x \notin g(\{x_n,\ n \in N\}),$$

the set

$$g(\{x_n,\ n \in N\} \cup \{x\})$$

is closed, and hence bicompact, and x is its nonisolated point. Thus we have shown that x is a χ-point in βX, as required. The lemma is proved.

Let A be some set and let $\{A_\gamma,\ \gamma \in \Gamma\}$ be the set of all its subsets, where $\Gamma = \{\gamma\}$ is the set of indices of the corresponding card–

nesses. Recall that a system $\mathfrak{F} = \{A_\gamma, \gamma \in \Gamma'\}$, where $\Gamma' \subset \Gamma$, is called a proper filter if it is centered, together with each set every set containing it belongs to it, and the intersection of any two elements of the system is again an element of the system (see, for example, ([2]), Ch. I, § 6, no. 1). If $\mathfrak{F}$ is a filter, then denote

$$\Gamma(\mathfrak{F}) = \{\gamma \in \Gamma, A_\gamma \in \mathfrak{F}\}.$$

We shall call a subset $A' \subset A$ essential with respect to the filter $\mathfrak{F}$ if $A' \cap A_\gamma \neq \emptyset$ for each $\gamma \in \Gamma(\mathfrak{F})$. Let $\mathfrak{F}$ be a free proper filter, i.e. the intersection of all sets belonging to it is empty. Denote by $T(\mathfrak{F})$ the topological space whose points are the elements of the set A and the filter $\mathfrak{F}$ itself, and whose basic open sets are all subsets of the set A and the sets of the form $A_\gamma \cup \{\mathfrak{F}\}$, where $\gamma \in \Gamma(\mathfrak{F})$. If A' is a set essential with respect to $\mathfrak{F}$, then, as is not difficult to see,

$$[A']_{T(\mathfrak{F})} \supset \mathfrak{F}.$$

Moreover, if one considers the filter

$$\mathfrak{F}|A' = \{A_\gamma \cap A', \gamma \in \Gamma(\mathfrak{F})\},$$

then it is not hard to show that $[A']_{T(\mathfrak{F})}$ is homeomorphic to $T(\mathfrak{F}|A')$. Note that the space $T(\mathfrak{F})$ is normal.

Let $\{X_\alpha, \alpha \in A\}$ be a family of topological spaces and $\prod_{\alpha \in A} X_\alpha$ the Cartesian product of the sets X_α. As usual, by

$$\mathfrak{S} \prod_{\alpha \in A} X_\alpha$$

we shall denote the set of those points of $\prod_{\alpha \in A} X_\alpha$ in which only a finite set of coordinates differs from the coordinates of some fixed point. (In the case when the X_α are topological groups, following Pontryagin ([6]), we shall call

$$\mathfrak{S} \prod_{\alpha \in A} X_\alpha$$

the product, and

$$\prod_{\alpha \in A} X_\alpha$$

the full product of the groups X_α.)

Let $\mathfrak{F}$ be some free proper filter of subsets of the set of indices A. Consider all sets

$$U = \prod_{\alpha \notin A_\gamma} U_\alpha \times \prod_{\alpha \in A_\gamma} X_\alpha \subset \prod_{\alpha \in A} X_\alpha,$$

where $U_\alpha \subset X_\alpha$, the U_α are open subsets, and $\gamma \in \Gamma(\mathfrak{F})$. It is not difficult to see that the totality of all sets of this form satisfies all the axioms of a topological base and defines on $\prod_{\alpha \in A} X_\alpha$, and hence also on

$$\mathfrak{S} \prod_{\alpha \in A} X_\alpha,$$

a non-discrete topology, which we shall denote below by $\mathcal{T}(\mathfrak{F})$. Note that if the spaces X_α are completely regular, then their product in the topology $\mathcal{T}(\mathfrak{F})$ is completely regular.

Theorem 1. Let $A = \{\alpha\}$ be a set of indices, $\{X_\alpha, \alpha \in A\}$ a family of topological spaces; $\mathfrak{F}$ a free filter of subsets of the set A; $A' \subset A$ a subset essential with respect to $\mathfrak{F}$. Then, if

$$\mathfrak{S} \prod_{\alpha \in A} X_\alpha$$

in the topology $\mathcal{T}(\mathfrak{F})$ is dyadic, then $\mathfrak{F}|A'$ is a κ-point in the space $\beta T(\mathfrak{F}|A')$.

Proof. We shall assume that the set A' is well ordered. Denote by B the set of those $\beta \in A'$ for which

$$B_\beta = \{\alpha \in A', \alpha \leq \beta\} \in \mathfrak{F}|A',$$

or, what is the same,

$$B_\beta \cup (A \setminus A') \in \mathfrak{F}.$$

Note that $A \setminus B \notin \mathfrak{F}$, i.e. B is an essential set. Let

$$x_0 \in X = \mathfrak{S} \prod_{\alpha \in A} X_\alpha.$$

For each $\beta \in B$ choose $y_\beta \in X_\beta$, $U_\beta \subset X_\beta$, $V_\beta \subset X_\beta$ so that

$$y_\beta \in U_\beta, \qquad V_\beta \cap U_\beta = \emptyset, \qquad \pi_\beta(x_0) \in V_\beta,$$

where $\pi_\beta(x_0)$ is the projection of the point x_0 into the space X_β. Put

$$W_\beta = \{x \in X,\ \pi_\alpha(x) \in V_\alpha \text{ for } \alpha < \beta,\ \pi_\beta(x) \in U_\beta\}$$

and

$$W = \bigcup_{\beta \in B} W_\beta.$$

It is not difficult to see that $W_\alpha \cap W_\beta = \emptyset$ for $\alpha \neq \beta$, W_β is an open set and hence the set W is open. For each $\beta \in B$ denote by x_β the point of X such that

$$\pi_\alpha(x_\beta) = \pi_\alpha(x_0) \quad \text{for } \alpha \neq \beta, \qquad \pi_\beta(x_\beta) = y_\beta.$$

Obviously, $x_\beta \in W_\beta \subset W$. We shall show that

$$x_0 \in [W].$$

For this it suffices to show that

$$x_0 \in [\{x_\beta, \beta \in B\}].$$

Let $\gamma \in \Gamma(\mathfrak{F})$ and

$$U = \mathfrak{S} \prod_{\alpha \notin A_\gamma} V'_\alpha \times \mathfrak{S} \prod_{\alpha \in A_\gamma} X_\alpha$$

be an arbitrary basic ···

a neighborhood of the point x_0, where $V'_\alpha \ni \pi_\alpha(x_0)$. Then, as was noted, $B \cap A_\gamma \neq \emptyset$, i.e. for some $\beta \in B$ one has $\beta \in B \cap A_\gamma$, and, evidently, $x_\beta \in U$. By the arbitrariness of the choice of U it follows that $x_0 \in [\{x_\beta,\ \beta \in B\}] \subset [W]$. It is known that a canonical closed set of a dyadic space is again a dyadic space ([4]). But then $Z = W \cup \{x_0\}$ is a dyadic space as a dense subspace of the dyadic space $[W]$. Now we shall show that

$$F = \{x_\beta,\ \beta \in B\} \cup \{x_0\}$$

is an open continuous image of the set Z. Define the mapping $\varphi : Z \to F$ as follows: if $x \in W_\beta$, then $\varphi(x) = x_\beta$, and $\varphi(x_0) = x_0$. Observe that in the set F all points, except x_0, are isolated. Therefore the continuity of the mapping

φ at points $x \in W$ follows from the fact that, for each $\beta \in B$, the set $\varphi^{-1}(x_\beta)$ is open. Let now

$$U = \left(\prod_{\alpha \notin A_\gamma} V'_\alpha \times \prod_{\alpha \in A_\gamma} X_\alpha \right) \cap F$$

be some basic neighborhood of the point x_0 in F.

Then put

$$U' = \prod_{\alpha \notin A_\gamma} (V'_\alpha \cap V_\alpha) \times \prod_{\alpha \in A_\gamma} X_\alpha.$$

Obviously, U' is also a neighborhood of x_0. We shall prove that $\varphi(U' \cap W) \subset U$. Let $x \in U' \cap W$; then $x \in U' \cap W_\beta$ for some $\beta \in B$. But $\pi_\beta(W_\beta) = U_\beta$, and

$$\pi_\beta(x) \in U_\beta \subset X_\beta \setminus V_\beta \subset X_\beta \setminus (V_\beta \cap V'_\beta),$$

whence it follows that $\beta \in A_\gamma$. Then also $\varphi(x) = x_\beta \in U$, which proves the continuity of the mapping φ. The openness of the mapping φ at points belonging to W is obvious, since the image of any point of W is an isolated point in F. To prove openness of φ at the point x_0, observe that the mapping φ on the set F is the identity, and for every neighborhood $U \ni x_0$ we have $\varphi(U) \supset F \cap U$, i.e. a neighborhood of the point x_0 in F. Thus we have shown that F is a continuous open, and consequently quotient, image of the dyadic space Z. Hence, by Lemma 1, the non-isolated point x_0 is a χ-point in the space βF. We shall prove now that F is homeomorphic to $T(\mathfrak{F}|B)$. Define the mapping $f : F \to T(\mathfrak{F}|B)$ as follows: $f(x_\beta) = \beta$, if $\beta \in B$, and $f(x_0) = \mathfrak{F}|B$. The continuity of the mapping f at the isolated points x_β is obvious. We may assume that a basic neighborhood of the point x_0 in F has the form

$$U = \left(\prod_{\alpha \notin A_\gamma} (V'_\alpha \cap V_\alpha) \times \prod_{\alpha \in A_\gamma} X_\alpha \right) \cap F.$$

But for $\beta \notin A_\gamma$, by definition,

$$\pi_\beta(x_\beta) = y_\beta \in X_\beta \setminus V_\beta,$$

i.e. $x_\beta \notin U$, and on the other hand, if $\beta \in A_\gamma$, then for $\alpha \notin A_\gamma$ we have

$$\pi_\alpha(x_\beta) = \pi_\alpha(x_0) \in V'_\alpha \cap V_\alpha,$$

thereby

$$U = \{x_\beta,\ \beta \in A_\gamma\} \cup \{x_0\}$$

and

$$f(U) = A_\gamma \cup \{\mathfrak{F}|B\}$$

is an open set. Conversely,

$$f^{-1}((A_\gamma \cap B) \cup \{\mathfrak{F}|B\}) = \{x_\beta,\ \beta \in A_\gamma\} \cup \{x_0\} = \left(\prod_{\alpha \notin A_\gamma} V_\alpha \times \prod_{\alpha \in A_\gamma} X_\alpha \right) \cap F$$

for each $\gamma \in \Gamma(\mathfrak{F})$, i.e. the preimage of an open set is open. Thus we have proved that f is an open continuous one-to-one mapping, i.e. a homeomorphism, and the point $\mathfrak{F}|B$ is a χ-point of the space $\beta T(\mathfrak{F}|B)$. But, as we noted, $T(\mathfrak{F}|B)$ is topologically embedded in $T(\mathfrak{F}|A')$, moreover as a closed subset, since every nonclosed subset in $T(\mathfrak{F})$ is discrete in itself. But the space $T(\mathfrak{F})$ is normal, and

$$[B]_{\beta T(\mathfrak{F})} = [T(\mathfrak{F}|B)]_{\beta T(\mathfrak{F}|A')} = \beta T(\mathfrak{F}|B);$$

as we have established, the point $\mathfrak{F}|B$ is a χ-point of the space $\beta T(\mathfrak{F}|B)$. Obviously, under the embedding of $T(\mathfrak{F}|B)$ in $T(\mathfrak{F}|A')$ the point $\mathfrak{F}|B$ passes to the point $\mathfrak{F}|A'$, and, consequently, $\mathfrak{F}|A'$ is a χ-point in the space $\beta T(\mathfrak{F}|A')$, as required. The theorem is proved.

Corollary 1. If the space $\prod_{\alpha\in A} X_\alpha$ is dyadic in the topology $\mathfrak{T}(\mathfrak{F})$, then $\mathfrak{F}$ is a χ-point in the space $\beta T(\mathfrak{F})$.

Let $\mathfrak{F}$ be a filter of subsets of some set A. A family $\{A_\gamma,\ \gamma \in \Gamma'\}$, where $\Gamma' \subset \Gamma(\mathfrak{F})$, will be called a quasibase of the filter $\mathfrak{F}$ if for every $\gamma_0 \in \Gamma(\mathfrak{F})$ the inequality

$$\inf_{\gamma\in\Gamma'} |A_\gamma \setminus A_{\gamma_0}| < \aleph_0$$

holds.

Theorem 2. Let $\{X_\alpha\}$ be a family of dyadic topological spaces and $\mathfrak{F}$ a free filter of subsets of the set of indices, and suppose that for every $\gamma \in \Gamma(\mathfrak{F})$ the space $\prod_{\alpha\in A\setminus A_\gamma} X_\alpha$ in the strong topology is dy-

adic. Then, if the filter $\mathfrak{F}$ has a countable quasi-base, the product $\bigotimes_{\alpha\in A} X_\alpha$ in the topology $\mathcal{T}(\mathfrak{F})$ is a dyadic space.

Theorem 2′. The assertion of Theorem 2 remains true if in its formulation, instead of the product $\bigotimes_{\alpha\in A} X_\alpha$, one speaks of the product $\prod_{\alpha\in A} X_\alpha$.

Let us formulate several remarks.

1. It is not difficult to verify that if $\{X_\alpha\}$ are topological groups, then for every free filter $\mathfrak{F}$ the topology $\mathcal{T}(\mathfrak{F})$ satisfies all the axioms of a topological group ([6]) and turns $\prod_{\alpha\in A} X_\alpha$, and hence also $\bigotimes_{\alpha\in A} X_\alpha$, into topological groups.
2. If the cardinality of each of the spaces X_α does not exceed $\aleph_0$, and the cardinality of the index set is also $\aleph_0$, then the product $\bigotimes_{\alpha\in A} X_\alpha$ also has cardinality $\aleph_0$.
3. It is obvious that for every infinite set A there exists a free filter $\mathfrak{F}$ of its subsets which is not an $\varkappa$-point of the space $\beta T(\mathfrak{F})$ (for example, every ultrafilter is such).

In view of these remarks and Corollary 1, we have

Theorem 3. There exist countable and, consequently, finally compact topological groups whose spaces are not dyadic.

Until now it has been unknown whether there exist extremely disconnected nondiscrete topological groups, but the following holds.

Theorem 4. Every nondiscrete dyadic topological space is not extremely disconnected.

Proof. As is known, the Čech extension of an extremely disconnected space is an extremely disconnected bicompactum ([3]); in extremely disconnected bicompacta there are no $\varkappa$-points ([7]); on the other hand, by Lemma 1, in the Čech extension of every nondiscrete dyadic topological space there are $\varkappa$-points, since in general there exist nonisolated ones. The contradiction obtained proves the assertion of the theorem. The theorem is proved.

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Note added in proof. Recently the author has proved (assuming the continuum hypothesis, $\exp \aleph_0 = \aleph_1$) the existence of countable topological groups whose spaces are nondiscrete and at the same time extremely disconnected.

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Note: Figure translations are in progress. See original paper for figures.