FACTOR IMAGES OF METRIC SPACES

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 513.83

MATHEMATICS

S. I. NEDEV

FACTOR IMAGES OF METRIC SPACES

(Presented by Academician P. S. Aleksandrov on 24 IV 1968)

Definition 1. A topological space X is called **generally metrizable** if on the set $X \times X$ one can define a nonnegative function $\rho(x, y)$ with the following properties: 1) $\rho(x, y) = 0$ if and only if $x = y$; 2) if $U \subset X$, then U is open in X if and only if for every point $x \in U$ there exists $\varepsilon > 0$ such that $U \supset O_\varepsilon^\rho x = E\{y : \rho(x, y) < \varepsilon\}$. The function ρ is called a **generalized metric**. If, in addition to 1) and 2), the function ρ also satisfies the condition: 3) $\rho(x, y) = \rho(y, x)$, then it is called a **symmetric**, and the space X is symmetrizable.

Let $f : X \to Y$ be a given mapping of a metric space X onto a set Y. If ρ is a metric generating the topology of the space X, then on the set Y one can introduce the symmetric $d(y_1, y_2) = \rho(f^{-1}y_1, f^{-1}y_2)$ and consider the topological space $Y_{\rho,f}$ with the topology generated (in the sense of item 2 of Definition 1) by the symmetric d. In ([1]), p. 144, A. V. Arkhangel' skii posed the problem of studying all topologies $Y_{\rho,f}$ that can be obtained by varying ρ (of course, without changing the topology of the space X). Theorem 4 of the present article shows that it may be of interest to vary ρ not only in the class of all metrics, but also in the class of all generalized metrics generating the given topology on X. Theorem 5 indicates a sufficiently broad class of mappings for which the symmetrizability of the preimage implies the metrizability of the image; the first three theorems give a characterization of symmetrizable and generally metrizable spaces in the spirit of ([3]).

Theorem 1. A T_1-space Y is generally metrizable if and only if there exist: a) a metric space X; b) its subspace X'; and c) a continuous mapping $f : X \to Y$ such that 1) $fX' = Y$; 2) if $U \subset Y$, $U = f(G) = f(G \cap X')$, where G is open in X, then U is open in Y.

Remark. The mapping f is then a quotient mapping.

Proof of Theorem 1. Necessity. Let Y be generally metrizable by the generalized metric ρ. For each natural number n, denote by A_n the discrete space equipotent to Y. Let X be the subspace of the space

$$\prod_{n=1}^{\infty} A_n,$$

consisting of all points $x = \{y_n\}_{n=1}^{\infty}$, for each of which there is a point $y_x \in Y$ such that: 1) $y_x \in \bigcap_{n=1}^{\infty} O^{\rho}_{1/n} y_n$; 2) to each natural number n one can assign an m such that, if $k \geq m$, then $O^{\rho}_{1/n} y_x \supset O^{\rho}_{1/k} y_k$. Let X' be the subspace of X consisting of all points all of whose coordinates are equal to one another.

For each $x \in X$, put $fx = y_x$. Since Y is assumed to be a T_1-space, f is a single-valued mapping. We show that f is continuous. Suppose that in X we have

$$x = \lim_{n \to \infty} x_n, \quad \text{where } x = \{y^n\}_{n=1}^{\infty}, \quad x_k = \{y_{ki}\}_{i=1}^{\infty}$$

$(k = 1, 2, \ldots)$, $fx = y$, $fx_k = y_k$. Let $\varepsilon > 0$. By assumption there exists m such that, if $n \geq m$, then $O^{\rho}_{\varepsilon} y \supset O^{\rho}_{1/n} y^n$. Moreover, since $\lim_{n\to\infty} x_n = x$, there exists l such that, if $n \geq l$, then $y_{nm} = y^m$. Then, for $n \geq \max(m, l)$,

$$y_n \in O^{\rho}_{1/m}(y_{nm}) = O^{\rho}_{1/m} y^m \subset O^{\rho}_{\varepsilon} y.$$

Let $U \subset Y$, $U = f(G) = f(G \cap X')$, where G is open in X. Suppose that U is not open in Y, and let $y_0 \in U$ be a point such that for every natural n

$$O^{\rho}_{1/n} y_0 \cap (Y \setminus U) \neq \Lambda.$$

If $y_n \in O^{\rho}_{1/n} y_0 \cap (Y \setminus U)$, then the point

$$x_k\{y_0, \underbrace{y_0, \ldots, y_0}_{k-1 \text{ times}}, y_k, y_k, \ldots, y_k, \ldots\}$$

does not belong to the set $f^{-1}U$, and hence, a fortiori, to the set G, since $fx_k = y_k \in Y \setminus U$; on the other hand, if $x_0 = \{y_0, y_0, \ldots, \ldots, y_0, \ldots\}$, then $x_0 \in G$, and, since $p(x_0, x_k) \leq 1/2^*$, for some k one must have $x_k \in G$. The contradiction obtained shows that U is open.

We proceed to prove the **sufficiency** of the condition of Theorem 1. For each point $y \in Y$ choose one point $x_y \in f^{-1}y \cap X'$ and set $Q_{ny} = fO^{\rho}_{1/n} x_y$. It is clear that, if $U \subset Y$ and U is open, then for any point $y \in U$ there is an $n = n(y, U)$ for which $Q_{ny} \subset U$. Conversely, suppose that for any point $y \in U$ there exists $n = n(y, U)$ for which $Q_{ny} \subset U$. Then

$$f^{-1}U \supset G = \bigcup_{y \in U} O^{\rho}_{1/n(y,u)} x_y, \qquad f(G \cap X') = U.$$

Consequently, by the condition of the theorem, U is open. Thus it is proved that Y satisfies the weak first axiom of countability (see ([1]), Definitions 2, 3), and the latter, as is easy to verify, is equivalent to the o-metrizability of the space Y (see, for example, ([4]), Theorem 1).

Theorem 2. A T_1-space Y is symmetrizable if and only if there exist: a) a metric space X; b) its subspace X', and c) a continuous mapping $f : X \to Y$

such that: 1) $fX' = Y$; 2) if $U \subset Y$ and $U = f(G) = f(G \cap X')$, where G is open in X, then U is open in Y; and 3) if U is open in Y and $y \in U$, then for some $\varepsilon > 0$

$$X' \cap O^\rho_\varepsilon f^{-1} y \subset f^{-1} U.$$

Proof. Having Theorem 1, to prove the necessity of Theorem 2 it suffices to make sure that, if the o-metric ρ is a symmetric, then for any set U open in Y and for any point $y_0 \in U$ there is an $\varepsilon = \varepsilon(y_0, U) > 0$ such that $O^\rho_\varepsilon f^{-1} y \cap X' \subset f^{-1} U$, where X, X', and f are taken from the proof of the necessity of Theorem 1. Since U is assumed open, there exists $\varepsilon > 0$ such that $O^\rho_\varepsilon y_0 \subset U$. Then, if $1/(k+1) \leq \varepsilon < 1/k$ and $\eta = 1/2^{k+1}$, it follows from $x \in O^\rho_\eta f^{-1} y \cap X'$ that $x = \{y, y, ...\}$ and that there exist $\{y_{k+i}\}_{i=2}^\infty$ for which

$$f\{y, y, \ldots, y, \underbrace{y}_{k+1 \text{ times}}, y_{k+2}, y_{k+3}, \ldots\} = y_0.$$

Consequently, $y_0 \in O^\rho_{1/(k+1)} y$, i.e. $y \in O^\rho_\varepsilon y_0 \subset U$.

To prove the sufficiency, we turn to the proof of sufficiency of the condition of Theorem 1. We show that the sets Q_{ny} constructed for any point $y \in Y$ satisfy the condition: from

$$y \in \bigcap_{n=1}^\infty Q_n y_n$$

it follows that $y = \lim y_n$. Indeed, if U is open and $y \in U$, then there exists k such that for any $n \geq k$

$$O^\rho_{1/n} f^{-1} y \cap X' \subset f^{-1} U.$$

Then, since the condition $y \in Q_{ny} n$ is equivalent to the condition

$$f^{-1} y \cap O^\rho_{1/n} x_{y_n} \neq \Lambda,$$

we obtain

$$x_{y_n} \in O^\rho_{1/n} f^{-1} y \cap X' \subset f^{-1} U,$$

i.e. $y_n \in U$. After this, the proof of Theorem 2 can be completed by reference to Theorem 3 of ([4]).

Definition 2 (see ([1]), p. 140, or 3, def. 1). A mapping $f : X \to Y$ of a space X, o-metrized by an o-metric ρ, onto a topological space Y is called a P-mapping if, for every open in

---

* For any two points $x = \{x_n\}_{n=1}^\infty \in X$ and $z = \{z_n\}_{n=1}^\infty \in X$, $p(x, z) = 1/2^k$, where

$$k = \min\{n : x_n \neq z_n\}.$$

of the set U and for any point $y \in U$ there is an $\varepsilon > 0$ such that $O^\rho_\varepsilon f^{-1} y \subset f^{-1} U$.

Theorem 3. A T_1-space Y is symmetrizable by a symmetric satisfying condition (AH)* if and only if there exist: a) a metric space X; b) its subspace X' and c) a continuous mapping $f : X \to Y$ such that: 1) $fX' = Y$; 2) if $U \subset Y$, $U = f(G) = f(G \cap X')$, where G is open in X, then U is open in Y, and 3) f is a Π-mapping (with respect to some metric generating the topology of the space X).

Proof. Necessity. Let the space Y be symmetrizable by a symmetric ρ satisfying condition (AH). To any point $y \in Y$ and to each natural number n we assign the set $Q_{ny} = O^{\rho}_{1/n}y$, such that $\operatorname{diam}(O^{\rho}_{1/(m-1)}y) \geq 1/n$, and $\operatorname{diam} Q_{ny} < 1/n$. For each natural n, let A_n denote the discrete space of the same cardinality as Y, and let X be the subspace of the space

$$\prod A_n,$$

consisting of all points $x = \{y_1, y_2, \dots, y_n, \dots\}$ for which there exist points $y_x \in Y$ such that $y_x \in \bigcap_1^\infty Q_{ny}n$. Finally, let X' be the subspace of X consisting of all points with constant coordinates. For each point $x \in X$ put $fx = y_x$. By arguments analogous to those used in the proof of Theorem 1, it is easy to show that the mapping f satisfies the first two conditions of Theorem 3.

Let us show that f also satisfies the third condition. Let U be open in Y and $y_0 \in U$. Then there exists a natural number n such that $O^{\rho}_{1/n}y_0 \subset U$. If $x \in O^{\rho}_{1/2}f^{-1}y_0$, then there exist $\{y^i\}_{i=n+1}^\infty$ such that, if $x = \{y_k\}_{k=1}^\infty$, then

$$f(y_1, y_2, \dots, y_n, y^{n+1}, y^{n+2}, \dots) = y_0.$$

Consequently, $y_0 \in Q_{ny}n$, $fx \in Q_{ny}n$, i.e. $\rho(y_0, fx) < 1/n$, whence $fx \in O^{\rho}_{1/n}y_0 \subset U$. Thus the necessity of the condition of Theorem 3 is proved.

Sufficiency. For any point $y \in Y$ choose a point $x_y \in f^{-1}y \cap X'$ and put $Q_{ny} = fO^{\rho}_{1/n}x_y$. Put also $\omega_n = \{Q_{ny}\}_{y\in Y}$ and $Q'_n y = St_{\omega_n}y$. On the set Y consider the following two o-metrics:

$$\rho(y_1, y_2) = \begin{cases} 2, & \text{if } y_2 \notin Q_1 y_1, \\ 1/2^n, & \text{if } y_2 \in Q_{ny}1/Q_{n+1}y_1, \\ 0, & \text{if } y_1 = y_2; \end{cases}$$

$$\rho'(y_1, y_2) = \begin{cases} 2, & \text{if } y_2 \notin Q'_1 y_1, \\ 1/2^n, & \text{if } y_2 \in Q'_{ny}1/Q'_{n+1}y_1, \\ 0, & \text{if } y_1 = y_2. \end{cases}$$

It is clear that the space Y is o-metrizable by the o-metric ρ and that the o-metric ρ' is a symmetric. We show that Y is symmetrizable by the symmetric ρ'. Indeed, let U be open in Y and let $y_0 \in U$. For some n, $O^{\rho}_{1/n}f^{-1}y \subset f^{-1}U$. Let $y \in O^{\rho'}_{1/4^n}y_0$. Consequently, $y \in Q_{2n}z$, where z is such that $y_0 \in Q_{2n}z$. The latter inclusion gives $O^{\rho}_{1/2n}x_z \cap f^{-1}y_0 \neq \Lambda$, whence we conclude that $O^{\rho}_{1/2n}x_z \subset O^{\rho}_{1/n}f^{-1}y_0$. Consequently, $f^{-1}y \cap O^{\rho}_{1/n}f^{-1}y_0 \neq \Lambda$, which gives $y \in U$. If, conversely, U contains each of its points together with some spherical neighborhood with respect to ρ', then U is open, since then U contains each of its points together with some spherical neighborhood with respect to ρ. Thus, the space Y is symmetrizable by the symmetric ρ'. Let now a sequence of points

* See ([1]), p. 147. A symmetric ρ satisfies condition (AH) if, whatever the point $x \in X$ and $\varepsilon > 0$, there exist

$\{y_n\}_{n=1}^{\infty}$ converges to the point y, and let $\varepsilon > 0$. For sufficiently large n we have $\rho(y, y_n) < 1/2^k \leq \varepsilon$, i.e., $y_n \in Q_{k+1}y$, and, consequently, for such (sufficiently large) m and n, $\rho'(y_m, y_n) < \varepsilon$. All this means that the semimetric ρ' satisfies condition (AH).

Theorem 4. A T_1-space Y is o-metrizable if and only if there exist: a) a metric space X; b) an o-metric $\tilde{p}$ on X consistent with the topology of X; and c) a quotient mapping $f : X \to Y$ which is a Π-mapping with respect to $\tilde{p}$. Moreover, if Y is o-metrizable, then for any of its o-metrics ρ there exist corresponding $X = X_\rho$, $f = f_\rho$, $\tilde{p} = \tilde{p}_\rho$ such that

$$\rho(y_1, y_2) = \tilde{p}(f^{-1}y_1, f^{-1}y_2)$$

for any two points $y_1, y_2 \in Y$.

Proof. Necessity. Let Y be o-metrizable with o-metric ρ, and let X, its metric p, and the mapping f be the same as in the proof of Theorem 1. Consider on X the o-metric $\tilde{p}$ defined as follows:

$$\tilde{p}(x_1, x_2) = p(x_1, x_2) - p(f^{-1}fx_1, f^{-1}fx_2) + \rho(fx_1, fx_2).$$

We first show that $\tilde{p}$ generates on X the same topology as p. Let

$$\lim_{n\to\infty} p(x_0, x_n) = 0.$$

Then

$$\lim_{n\to\infty} p(f^{-1}fx_0, f^{-1}fx_n) = 0,$$

since

$$p(f^{-1}fx_0, f^{-1}fx_n) \leq p(x_0, x_n),$$

and

$$\lim_{n\to\infty} \rho(fx_0, fx_n) = 0,$$

since f is continuous. Hence

$$\lim_{n\to\infty} \tilde{p}(x_0, x_n) = 0.$$

Conversely, let

$$\lim_{n\to\infty} \tilde{p}(x_0, x_n) = 0.$$

Then

$$\lim_{n\to\infty} \rho(fx_0, fx_n) = 0,$$

and hence, by the definition of the mapping f,

$$\lim_{n\to\infty} p(f^{-1}fx_0, f^{-1}fx_n) = 0.$$

Therefore,

$$\lim_{n\to\infty} p(x_0, x_n) = 0.$$

The fact that $\rho(y_1, y_2) = \tilde{p}(f^{-1}y_1, f^{-1}y_2)$ and that f is a Π-mapping with respect to $\tilde{p}$ follows directly from the definition of $\tilde{p}$.

Sufficiency. Let X be o-metrizable with o-metric $\tilde{p}$, and let the mapping $f : X \to Y$ be quotient and, with respect to $\tilde{p}$, a Π-mapping. On the set Y introduce the o-metric

$$\rho(y_1, y_2) = \tilde{p}(f^{-1}y_1, f^{-1}y_2)$$

and show that it generates the topology of the space Y. Indeed, if U is open in Y and $y_0 \in U$, then from

$$\varepsilon = \tilde{p}(f^{-1}y_0, X \setminus f^{-1}U) > 0$$

it follows that $O^{\rho}_{\varepsilon} y_0 \subset U$. Conversely, suppose that the set U contains all its points together with their spherical neighborhoods with respect to ρ, and let $x_0 \in f^{-1}U$. If

$$O^{\rho}_{\varepsilon} fx_0 \subset U,$$

then

$$O^{\tilde{p}}_{\varepsilon} x_0 \subset f^{-1}U,$$

and, consequently, $f^{-1}U$ is open in X. Since by assumption f is quotient, U is open in Y. The theorem is proved.

We note that if, in the proof of Theorem 4, the words "o-metrizable" and "o-metric $\tilde{p}$" are replaced by: a) "symmetrizable" and "symmetric $\tilde{p}$," or by b) "symmetrizable with a symmetric satisfying the weak Cauchy condition" and "a symmetric $\tilde{p}$ satisfying the weak Cauchy condition," then true assertions are obtained, which can be proved in the same way.

Definition 3. A mapping $f : X \to Y$ from an o-metrizable space X with o-metric ρ onto a topological space Y is called K-uniform (with respect to ρ) if for every bicompact set $\Phi \subset Y$ and every neighborhood U of it the condition

$$\rho(f^{-1}\Phi, X \setminus f^{-1}(U)) > 0$$

is satisfied.

Theorem 5. A T_2-space is metrizable if and only if it is a quotient K-uniform (with respect to some symmetric) image of a symmetrizable space.

The proof of Theorem 5 repeats exactly the arguments of A. V. Arhangel' skii given in the proof of Theorem 3.3 of ([1]).

Received
24 IV 1968

REFERENCES

[1] A. V. Arhangel' skii, *Uspekhi Mat. Nauk*, **21**, no. 4, 133 (1966).
[2] A. V. Arhangel' skii, *Dokl. Akad. Nauk SSSR*, **64**, 247 (1965).
[3] R. W. Heath, *Fund. Math.*, **57**, 1, 91 (1965).
[4] S. I. Nedev, *Dokl. Bulgarsk. Akad. Nauk*, **20**, no. 6, 513 (1967).

* See ([2]). A symmetric of a space X satisfies the weak Cauchy condition if, from the fact that A is not closed in X, it follows that for every $\varepsilon > 0$ there are two points x and y in A such that $\rho(x, y) < \varepsilon$ and $x \neq y$.

Note: Figure translations are in progress. See original paper for figures.