ON BICOMPACT SEMIREGULAR AND HAUSDORFF EXTENSIONS

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Abstract

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The purpose of this paper is to obtain all semiregular (= T_b)-bicompact extensions of T_λ-spaces * (Theorem 1) and all Hausdorff bicompact extensions of Tikhonov (= completely regular) spaces (Theorem 2) in the form of limits of certain projection spectra, namely spectra that are so-called refinements (maximal finite ones) of the spectrum S_X of the given space X.

Definition 1. A projection spectrum $S^* = \{K_\alpha^*, \mathfrak{F}_\alpha^{\alpha'}\}$ is called a **refinement** of the spectrum $S = \{K_\alpha, \mathfrak{F}_\alpha^{\alpha'}\}$ if both spectra are directed by one and the same set of indices $\{\alpha\}$, and for each α a one-to-one mapping f_α is given from the set $\dot{K}_\alpha$ of all vertices of the complex K_α onto the set $\dot{K}_\alpha^*$ of all vertices of the complex K_α^*, inducing a simplicial mapping f_α of the complex K_α into the complex K_α^*, commuting with the projections in the sense that, for $\alpha' \geq \alpha$, we have

$$f_\alpha \mathfrak{F}_\alpha^{\alpha'} = \mathfrak{F}_\alpha^{*\alpha'} f_{\alpha'};$$

if, moreover, for all α one has $f_\alpha K_\alpha = K_\alpha^*$, then the spectra S and S^* are isomorphic to one another.

Remark 1. Let the spectrum $S^* = \{K_\alpha^*, \mathfrak{F}_\alpha^{*\alpha'}\}$ be a refinement of the spectrum $S = \{K_\alpha, \mathfrak{F}_\alpha^{\alpha'}\}$. Identify each vertex e_α of the complex K_α with the corresponding vertex $f_\alpha e_\alpha$ of the complex K_α^*; then the complexes K_α and K_α^* have one and the same set of vertices, hence $K_\alpha \subseteq K_\alpha^*$ and $\mathfrak{F}_\alpha^{*\alpha'} = \mathfrak{F}_\alpha^{\alpha'}$.

Let the spectrum $S^{(1)} = \{K_\alpha^{(1)}, \mathfrak{F}_\alpha^{\alpha'}\}$ be a refinement of the spectrum $S^{(2)} = \{K_\alpha^{(2)}, \mathfrak{F}_\alpha^{\alpha'}\}$. Then every maximal thread of the spectrum $S^{(2)}$ is a thread of the spectrum $S^{(1)}$; if it is also maximal in the spectrum $S^{(1)}$, then we call it **invariant under the refinement** $S^{(1)}$ of the spectrum $S^{(2)}$.

Let $S_X = \{N_\alpha, \mathfrak{F}_\alpha^{\alpha'}\}$ be the spectrum of the T_λ-space X. If $x \in X$, then by $t_\alpha(x)$ we denote the simplex of the nerve N_α whose support corresponds to the set of all elements of the covering α containing the point x. It is shown that $\{t_\alpha(x)\}$ is a maximal thread $\xi(x) = \{t_\alpha(x)\}$ of the spectrum S_X.

Definition 2. A refinement $S = \{N'_\alpha, \mathfrak{F}_\alpha^{\alpha'}\}$ of the spectrum $S_X = \{N_\alpha, \mathfrak{F}_\alpha^{\alpha'}\}$ of a given T_λ-space X is called **correct** if, under this refinement, every thread of the form $\xi = \{t_\alpha(x)\}$ of the spectrum S_X is invariant.

Let $S^{(1)} = \{K_\alpha^{(1)}, \mathfrak{F}_\alpha^{\alpha'}\}$ and $S^{(2)} = \{K_\alpha^{(2)}, \mathfrak{F}_\alpha^{\alpha'}\}$ be two refinements of the spectrum $S = \{K_\alpha, \mathfrak{F}_\alpha^{\alpha'}\}$. We shall say that $S^{(1)}$ is a natural refinement of the spectrum $S^{(2)}$ if, for all α, we have $K_\alpha^{(2)} \subseteq K_\alpha^{(1)}$.

Remark 2. The set of all correct refinements of the spectrum S_X is partially ordered: $S^{(2)}$ follows $S^{(1)}$ if $S^{(1)}$ is a natural refinement of the spectrum $S^{(2)}$. The set of all bicompact semiregular extensions of a given T_λ-space X is partially ordered in the usual way: set $b_2X \geq b_1X$ if there exists a continuous mapping of the space b_2X onto b_1X with fixed $x \in X$.

* For terminology and notation see (²). We consider only finite (= consisting of finite complexes) spectra. Every finite spectrum is bounded (= every one of its threads is contained in a maximal thread) (³), and, consequently, complete (¹).

2. Let $S = \{K_\alpha, \mathfrak{G}_\alpha^{\alpha'}\}$ be an arbitrary spectrum and let the simplex $t_{\alpha_\sigma} \in K_{\alpha_\sigma}$. Recall that by Ot_{α_σ}, respectively Φt_{α_σ}, we denote the set of all maximal threads $\xi' = \{t'_\alpha\}$ satisfying the condition $t'_\alpha \leq t_{\alpha_\sigma}$, respectively $t'_\alpha \geq t_{\alpha_\sigma}$. By the definition of the topology in the inverse-limit space $\tilde{S}$ of the spectrum S, the sets Ot_α form an open base of the space $\tilde{S}$. We shall call the spectrum S **symmetric** if the sets Φt_α form a closed base of the space; here it is enough to restrict oneself to the zero-dimensional simplexes $t_\alpha = e_\alpha$.

Remark 3. If the spectrum S_X for a given space X exists, then it is symmetric (([^2], Theorem 1).

Definition 3. A spectrum S is called a **λ-strengthening** of the spectrum S_X of a given T_λ-space X if it is symmetric and is a correct strengthening of the spectrum S_X.

Theorem 1. *Let X be an arbitrary T_λ-space. For every bicompact semiregular extension $\bar{X} = bX$ of the space X, the spectrum $S_{\bar{X}}$ is a λ-strengthening of the spectrum S_X. Conversely, the limit of every λ-strengthening S of the spectrum S_X is a bicompact semiregular extension of the space X.*

The correspondence $bX \to S_{bX}$ between all bicompact semiregular extensions bX of the space X and all λ-strengthenings of the spectrum S_X of this space, obtained by virtue of what has just been said, is one-to-one and preserves the order in the sense that from $b_2X \geq b_1X$ it follows that $S_{b_2X} \geq S_{b_1X}$.

We prove the first assertion of Theorem 1; we take the following proposition as known:

(A) *Let $\bar{X}$ be an arbitrary extension of the space X. Then, denoting by A any canonical closed (we write briefly $\varkappa a$) set of the space X, and by $\bar{A}$ its closure in $\bar{X}$, we have $X \cap \bar{A} = A$, and the correspondence $A \to \bar{A}$ is a one-to-one correspondence between all $\varkappa a$-sets in X and all $\varkappa a$-sets in $\bar{X}$, and hence also a one-to-one correspondence between all decompositions $\alpha = \{A_\alpha^1, \dots, A_\alpha^{s_\alpha}\}$ of the space X and all decompositions $\bar{\alpha} = \{\bar{A}_\alpha^1, \dots, \bar{A}_\alpha^{s_\alpha}\}$ of the space $\bar{X}$, which is an* *order isomorphism (the relations $\alpha' > \alpha$ and $\bar{\alpha}' > \bar{\alpha}$ are equivalent). Hence it follows at once that*

Lemma 1. *If the spectrum S_X exists, then the spectrum $S_{\bar{X}}$ also exists, and it is a strengthening of the spectrum S_X.*

Lemma 2. *If $\bar{X}$ is a T_λ-extension of the space X, then $S_{\bar{X}}$ is a correct strengthening of the spectrum S_X.*

This lemma follows easily from the following assertion:

(B) *Suppose that, among the elements of a decomposition $\alpha = \{A^1_\alpha, \dots, A^{s_\alpha}_\alpha\}$ of the space X, a given point $x \in X$ is contained in the elements $A^{i_0}_\alpha, \dots, A^{i_r}_\alpha$, and only in them. Then, among the elements $\bar{A}_i$ of the corresponding decomposition $\bar{\alpha}$, only $\bar{A}^{i_0}_\alpha, \dots, \bar{A}^{i_r}_\alpha$ contain the point x.*

Indeed, from $x \in \bar{A}^i_\alpha$ it follows that $x \in X \cap \bar{A}^i_\alpha = A^i_\alpha$.

Let $\bar{X}$ be a bicompact T_ξ (hence T_λ)-extension of the space X. Relying on Lemma 2 and Remark 3, we conclude that the spectrum $S_{\bar{X}}$ is a λ-strengthening of the spectrum S_X.

The correspondence $bX \to S_{bX}$ is one-to-one. Indeed, let the spectra S_{b_1X} and S_{b_2X} be naturally isomorphic. Since $\tilde{S}_{b_1X} = b_1X$, $\tilde{S}_{b_2X} = b_2X$ (see [^2]), it follows that $b_1X = b_2X$.

We pass to the second part of Theorem 1.

Lemma 3. *Let X be an arbitrary T_λ-space and let $S = \{N_\alpha, \mathfrak{G}^{\alpha'}_\alpha\}$ be a correct strengthening of the spectrum S_X. Then the limit space $\tilde{S}$ is a bicompact (T_1)-extension of the space X, semiregular if S is a λ-strengthening.*

By Theorems 1 and 4 of [2], the space $\hat{S}_X$ is a bicompact T_ζ-extension of the space X. Denoting by $\hat{S}$ the full limiting space of the (arbitrary) spectrum S (consisting of all, and not only maximal, threads of the spectrum S, with the same topology as in $\hat{S}$), we have the natural mapping $f : \hat{S}_X \to \hat{S}$. From the correctness of the refinement S it follows easily that under the mapping f the set $X \subseteq \tilde{S}_X \subseteq \hat{S}_X$ is mapped topologically onto the set $fX \subseteq \tilde{S} \subseteq \hat{S}$. It remains to show that fX is dense in $\tilde{S}$.

Let Ost_α be an arbitrary element of the open base of the space $\tilde{S}$. Take some vertex e_α of the simplex $t_\alpha \in N'_\alpha$. Since X is dense in $\tilde{S}_X$, there exists a point $x \equiv \xi(x) \in Oe_\alpha \subseteq \tilde{S}_X$. Then $t_\alpha(x) = e_\alpha \leq t_\alpha$, so that (in view of the correctness of the refinement S) we have $\tilde{S} \ni x \equiv \xi(x) \in O_{\hat{S}} t_\alpha$, and fX is dense in $\tilde{S}$. Let S be a λ-refinement of the spectrum S_X. We shall prove that then the extension $\tilde{S}$ is semiregular. For this it is enough to prove that all $\Phi_S e_\alpha$ are α-sets. This in turn follows from the equality

$$\Phi_S e^i_\alpha = [A^i_\alpha]_{\tilde{S}}, \tag{1}$$

valid for every correct refinement S of the spectrum S_X (by e^i_α one always denotes the vertex of the nerve N_α corresponding to the element A^i_α of the cover α). Fix $\alpha = \alpha_0$. The inclusion $[A^i_{\alpha_0}]_{\tilde{S}} \subseteq \Phi_S e_{\alpha^i}$ is checked directly. We prove the reverse inclusion. Let $\xi = \{t_\alpha\} \in \Phi_S e^i_{\alpha_0} \subseteq \tilde{S}$. It is required, for arbitrary $\alpha = \alpha_1$, to find a point $x \equiv \xi(x) \equiv \{t_\alpha(x)\} \in A^i_{\alpha_0}$ satisfying the condition $t_{\alpha_1}(x) \leq t_{\alpha_1} \in \xi$. For this we take α', $\alpha' \geq \alpha_0, \alpha_1$. Then $\mathfrak{D}^{\alpha'}_\alpha \cdot t_{\alpha'} = t_{\alpha_0}$, $\mathfrak{D}^{\alpha'}_{\alpha_1} t_{\alpha'} = t_{\alpha_1}$.

Let $e^k_{\alpha'} \leq t_{\alpha'}$ be such a vertex that $\mathfrak{D}^{\alpha'}_\alpha e^k_{\alpha'} = e^i_{\alpha_0} \leq t_{\alpha_0}$; then $\mathfrak{D}^{\alpha'}_{\alpha_1} e^k_{\alpha'} = e^j_{\alpha_1} \leq t_{\alpha_1}$, and, consequently,

$$A^k_{\alpha'} \subseteq A^i_{\alpha_0}, \qquad A^k_{\alpha'} \subseteq A^j_{\alpha_1}, \qquad A^i_{\alpha_0} \cap I A^j_{\alpha_1} \neq \Lambda.$$

Take $x \in A^i_{\alpha_0} \cap I A^j_{\alpha_1}$, $\xi(x) = \{t_\alpha(x)\}$. Then $\xi(x) \equiv x \in A^i_{\alpha_0}$ and $t_{\alpha_1}(x) = e^j_{\alpha_1} \leq t_{\alpha_1}$, and the point $x \equiv \xi(x)$ we need has been found.

We shall show that the one-to-one mapping $bX \to S_{bX}$ is a mapping "onto." For this we prove the formula

$$S_{\tilde{S}} = S, \tag{2}$$

valid for every λ-refinement $S = \{N'_\alpha, \mathfrak{D}^{\alpha'}_\alpha\}$ of the spectrum $S_X = \{N_\alpha, \mathfrak{D}^{\alpha'}_\alpha\}$ of the given T_λ-space X. Since, by Lemma 3, $\tilde{S}$ is a bicompact T_ζ-extension of the space X, the spectrum $S_{\tilde{S}} = \{N''_\alpha, \mathfrak{D}^{\alpha'}_\alpha\}$ is a λ-refinement of the spectrum S_X. It remains to prove that the spectrum $S_{\tilde{S}} = \{N''_\alpha, \mathfrak{D}^{\alpha'}_\alpha\}$ is naturally isomorphic to S, which follows from the assertion

(B) The vertices $e^{i_0}_\alpha, \dots, e^{i_r}_\alpha$ of the nerve N_α then and only then are the vertices of some simplex t_α in the complex N'_α, when $[A^{i_0}_\alpha]_{\tilde{S}} \cap \dots \cap [A^{i_r}_\alpha]_{\tilde{S}} \neq \Lambda$.

Indeed, let $|e^{i_0}_\alpha, \dots, e^{i_r}_\alpha| = t_\alpha \in N'_\alpha$. Since the spectrum S is complete [4], $\Lambda \neq \Phi_S t_\alpha = \Phi_S e^{i_0}_\alpha \cap \dots \cap \Phi_S e^{i_r}_\alpha$ and $[A^{i_0}_\alpha]_{\tilde{S}} \cap \dots \cap [A^{i_r}_\alpha]_{\tilde{S}} \neq \Lambda$ by virtue of formula (1).

Conversely, let $\Lambda \neq [A^{i_0}_\alpha]_{\tilde{S}} \cap \dots \cap [A^{i_r}_\alpha]_{\tilde{S}} = \Phi_S e^{i_0}_\alpha \cap \dots \cap \Phi_S e^{i_r}_\alpha \ni \xi = \{t_\alpha\}$. Then the vertices $e^{i_0}_\alpha, \dots, e^{i_r}_\alpha$ are vertices of the simplex t_α, and the simplex $|e^{i_0}_\alpha \dots e^{i_r}_\alpha| \in N'_\alpha$ exists.

We prove that from $b_2X \geq b_1X$ it follows that $S_{b_2X} \geq S_{b_1X}$. But if $b_2X \geq b_1X$, then from $[A^{i_0}_\alpha]_{b_2X} \cap \dots \cap [A^{i_r}_\alpha]_{b_2X} \neq \Lambda$ it follows that $[A^{i_0}_\alpha]_{b_1X} \cap \dots \cap [A^{i_r}_\alpha]_{b_1X} \neq \Lambda$, and hence S_{b_1X} is a natural refinement of the spectrum S_{b_2X}.

Definition 4*. A thread $\xi = \{t_\alpha\}$ of a spectrum $S = \{K_\alpha, \mathfrak{D}^{\alpha'}_\alpha\}$ is called **regular** if for every α there exist α', $\alpha' \geq \alpha$, such that

* In [4], V. I. Ponomarev gave another definition of regularity, equivalent to ours for all complete spectra.

that every simplex $t'_{\alpha'} \in K_{\alpha'}$ having a common face with the simplex $t_\alpha \in \xi$ is projected into a face of the simplex $t_\alpha \in \xi$: $\delta_\alpha^{\alpha'} t_{\alpha'} \leq t_\alpha$.

A spectrum all of whose maximal threads are regular is called regular; its limit space is regular (Ponomarev).

A regular spectrum $S = \{N_\alpha, \delta_\alpha^{\alpha'}\}$ which is a correct strengthening of the spectrum S_X of a (Tychonoff) space X will be called an H-strengthening of the spectrum S_X. Its limit space $\tilde{S}$ is a bicompact T_2-extension of the space X. Conversely, the spectrum of every bicompact T_2-extension bX is regular (^{4}); hence, being, by the preceding results, a correct strengthening of the spectrum S_X, it is an H-strengthening. Just as in the case of bicompact T_ξ-extensions, one proves the mutual one-to-one character of the mapping $bX \to S_{bX}$ from the set of all bicompact T_2-extensions of the space X to the set of all H-strengthenings of the spectrum S_X.

This mapping is a "mapping onto," which follows from equality (2), proved for H-strengthenings in the same way as for λ-strengthenings (formula (2) is valid even for any correct strengthening).

Let us prove that the mapping under consideration is an order isomorphism. We shall call a thread of a spectrum **proper** if there exists a unique maximal thread containing it. A spectrum is proper if all its threads are proper.

Lemma 4. *Every H-strengthening $S = \{N'_\alpha, \delta_\alpha^{\alpha'}\}$ of the spectrum S_X of a Tychonoff space X is proper.*

Indeed, $\tilde{S}$ is a T_2-space; hence, for any two distinct points $\xi^1 = \{t_\alpha^1\} \in \tilde{S}$ and $\xi^2 = \{t_\alpha^2\} \in \tilde{S}$ there is an $\alpha = \alpha_0$ such that $O_{\alpha_0}\xi^1 \cap O_{\alpha_0}\xi^2 = \Lambda$. Lemma 4 follows from the fact that $t_{\alpha_0}^1 \in \xi^1$ and $t_{\alpha_0}^2 \in \xi^2$ do not have a single common vertex $e_{\alpha_0}^i$ (if $e_{\alpha_0}^i$ were such a vertex, then $O_{Se_{\alpha_0}^i} \subseteq O_{\alpha_0}\xi^1 \cap O_{\alpha_0}\xi^2$). But S is a correct strengthening of the spectrum S_X, and therefore $O_{Se_{\alpha_0}^i} \neq \Lambda$. A contradiction.

Lemma 5. *Let a proper spectrum $S_i = \{C_\alpha, \delta_\alpha^{\alpha'}\}$ be a strengthening of a (bounded) spectrum $S = \{K_\alpha, \delta_\alpha^{\alpha'}\}$. Then, assigning to each maximal thread $\xi = \{t_\alpha\} \in \tilde{S}$ the unique maximal thread containing it, $f\xi = \{t_\alpha^{(c)}\}$, we obtain a natural mapping $f : \tilde{S} \to \tilde{S}_c$ onto $\tilde{S}_c$. If the spectrum S_c is regular, then f is continuous.*

Let us prove that $f : \tilde{S} \to \tilde{S}_c$ is a mapping "onto." Take arbitrarily $\eta = \{t_\alpha^{(c)}\} \in \tilde{S}_c$ and in the spectrum $[\eta] = \{[t_\alpha^{(c)}], \delta_\alpha^{\alpha'}\}$, where by $[t_\alpha^{(c)}]$ is denoted the combinatorial closure of the simplex $t_\alpha^{(c)}$, take some zero-dimensional thread $\eta^0 = \{e_\alpha\}$. This thread is also a thread of the spectrum S, and it is contained in it in the maximal thread $\xi = \{t_\alpha\}$. Then $f\xi = \eta$ (otherwise the thread η^0 of the proper spectrum S_c would have two maximal threads containing it: η and $f\xi$). We leave the proof of the continuity of the mapping f to the reader. Thus, we have

Theorem 2. *Let X be an arbitrary Tychonoff space. To every bicompact Hausdorff extension bX of the space X there corresponds an H-strengthening S_{bX} of the spectrum S_X. Moreover, every H-strengthening S_{bX} of the spectrum S_X is put in correspondence with a unique Hausdorff bicompact extension $\tilde{S} = bX$ of the space X.*

The one-to-one correspondence $bX \to S_{bX}$ thus obtained between the two partially ordered sets—the set of all Hausdorff bicompact extensions of the Tychonoff space X and the set of all H-strengthenings of its spectrum S_X—is an order isomorphism.

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Note: Figure translations are in progress. See original paper for figures.

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