ON THE BOUNDARY-LAYER EQUATION

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Abstract

Full Text

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ON THE BOUNDARY-LAYER EQUATION

IN THE TWO-DIMENSIONAL THEORY OF OCEAN CURRENTS

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1. Introduction and statement of the problem. A widely used method for studying steady ocean currents is the analysis of depth-averaged flows (the method of total flows). As shown in (7), in this method, after passing to dimensionless coordinates, for the stream function one obtains the equation

$$\alpha\Delta^2\psi + \beta\left(\frac{\partial\psi}{\partial x_1}\frac{\partial\Delta\psi}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\Delta\psi}{\partial x_1}\right) + \frac{\partial\psi}{\partial x_1} - \operatorname{rot}_{x_3}\vec{\tau} = 0, \tag{1}$$

where $\vec{\tau}$ is the vector of the tangential wind stress, and α and β are positive constants expressed in terms of the main parameters of the flow. The stream function must satisfy no-slip conditions on the solid boundaries of the region filled with fluid, and on the liquid boundaries—the conditions of slip and no flow across the boundary.

This leads to the consideration of the following problem: in the domain Ω it is required to find a solution of the equation

$$\alpha\Delta^2\psi + \beta\left(\frac{\partial\psi}{\partial x_1}\frac{\partial\Delta\psi}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\Delta\psi}{\partial x_1}\right) + \frac{\partial\psi}{\partial x_1} = f(x_1, x_2), \tag{2}$$

satisfying the boundary conditions:

$$\begin{aligned} &\psi = 0, \qquad \partial\psi/\partial n = 0 \qquad \text{on the part of the boundary } \Gamma_0,\\ &\psi = 0, \qquad \partial^2\psi/\partial n^2 = 0 \quad \text{on the remaining part of the boundary } \Gamma_1. \end{aligned} \tag{3}$$

With respect to the boundary Γ of the domain Ω we make the following assumptions: it consists of a segment of the straight line $x_2 = 0$ (liquid boundary) and a piecewise smooth curve Γ_0, and it is assumed in advance that Γ_1 and Γ_0

intersect at nonzero angles not exceeding $\pi/2$, while the pieces of the curve Γ_0 intersect at nonzero angles not exceeding π.

In the present paper the existence and uniqueness of the solution of problem (2)–(3) is proved in the functional space $\overset{\circ}{W}{}_2^4$.

2. Functional spaces and inequalities. By $\mathcal{L}_p$ we shall denote the set of functions $\psi(x)$ summable with exponent p and with norm

$$\|\psi\|_{\mathcal{L}p} = \left(\int_\Omega |\psi(x)|^p \, dx \right)^{1/p} \qquad (p > 1).$$

The set of functions $\psi(x)$ $(x \in \Omega \cup \Gamma)$, l times continuously differentiable up to the boundary Γ, is denoted by $C^l(\Omega \cup \Gamma)$. The completion of the manifold $C^l(\Omega \cup \Gamma)$ in the norm

$$\|\psi\|_{W_p^l} = \sum_{|\alpha| \le l} \|D^\alpha \psi\|_{\mathcal{L}p} \qquad (p > 1) \tag{4}$$

is called the space of S. L. Sobolev and is denoted by W_p^l. Here $D^\alpha \psi = (D_1^{\alpha_1} D_2^{\alpha_2})\psi$, $D_j = \partial/\partial x_j$ $(j = 1, 2)$, $\alpha = (\alpha_1, \alpha_2)$ is a vector with integer nonnegative coordinates, $|\alpha| = \alpha_1 + \alpha_2$. The set of functions $\psi(x)$ belonging to $C^4(\Omega \cup \Gamma)$ and satisfying conditions (3) is denoted by $C_0^4(\Omega \cup \Gamma)$. The closure of $C_0^4(\Omega \cup \Gamma)$ in the norm

of the space $W_2^4(\Omega)$ will be a subspace of $W_2^4(\Omega)$, and we denote it by $\widetilde{W}_2^4(\Omega)$.

By $A\psi$ we denote the nonlinear part of equation (2).

Lemma 1. For all elements $\psi(x)$ belonging to $\widetilde{W}_2^4(\Omega)$, the equality

$$(A\psi, \psi)_{L_2} = 0. \tag{5}$$

holds.

If $\psi(x) \in C_0^4(\Omega \cup \Gamma)$, then equality (5) is verified by integration by parts, and for any function from $\widetilde{W}_2^4(\Omega)$ it is obtained by passage to the limit.

Lemma 2. The inequalities

$$|\psi(x)| \le 2\gamma \|\Delta\psi\|_{L_2} \qquad (x \in \Omega \cup \Gamma), \tag{6}$$

$$\|\partial\psi/\partial x_i\|_{L_2}^2 \le 6\gamma \|\Delta\psi\|_{L_2} \qquad (i = 1, 2), \tag{7}$$

hold, where γ is the diameter of the domain Ω, for all elements of the space $\widetilde{W}_2^4(\Omega)$.

We note that inequalities of the type (6), (7) follow from embedding theorems and the results of paper (3). We have obtained them anew in order to specify the constants.

Lemma 3. The problem

$$\Delta^2\psi = \varphi(x_1, x_2),$$
$$\psi|_\Gamma = 0, \qquad \partial\psi/\partial n|_{\Gamma_0} = 0, \qquad \partial^2\psi/\partial n^2|_{\Gamma_1} = 0$$

has a unique solution in the space $\widetilde{W}_2^4(\Omega)$ for any function $\varphi(x)$ belonging to $\mathcal{L}_2(\Omega)$, and moreover the inequality

$$\|\psi(x)\|_{W_2^4} \leq C\|\varphi\|_{L_2}, \tag{8}$$

holds, where C is a constant independent of $\psi(x)$.

Proof. Uniqueness follows from (6).

Denote by Ω^* the domain obtained by adjoining to Ω the domain Ω_1, symmetric with respect to the boundary Γ_1. Extend the function $\varphi(x_1, x_2)$ oddly with respect to x_2 to the domain Ω^*, and denote this extension by $\varphi^*(x_1, x_2)$. Consider the auxiliary problem

$$\Delta^2\psi^* = \varphi^*(x_1, x_2),$$
$$\psi^* = 0, \qquad \partial\varphi^*/\partial n = 0 \quad \text{on } \Gamma^*,$$

where Γ^* is the boundary of the domain Ω^*. From paper (2) it follows that for any $\varphi^*(x_1, x_2) \in \mathcal{L}_2(\Omega^*)$ there exists a unique solution ψ^* of the auxiliary problem in the space $W_2^4(\Omega^*)$, and moreover the inequality

$$\|\psi^*\|_{W_2^4(\Omega^*)} \leq C_1\|\varphi^*\|_{L_2(\Omega^*)}. \tag{9}$$

holds.

By virtue of the construction of the function $\varphi^*(x)$, the proof of the lemma follows from this.

It follows from Lemma 3 that the operator Δ^2, acting from $\widetilde{W}_2^4(\Omega)$ into the space $\mathcal{L}_2(\Omega)$, has an inverse linear bounded operator

$$B = (\Delta^2)^{-1}.$$

Then problem (2)–(3) can be reduced to an operator equation in the Hilbert space $\widetilde{W}_2^4(\Omega)$, namely to an equation of the form

$$\psi = P\psi, \tag{10}$$

where

$$P\psi = -\frac{1}{a}B\left(\beta A\psi + \frac{\partial\psi}{\partial x_1} - f(x)\right). \tag{11}$$

The operator P acts from $\widetilde{W}_2^4(\Omega)$ into $\widetilde{W}_2^4(\Omega)$, and, moreover:

Lemma 4. The operator P is completely continuous in the space $\widetilde{W}_2^4(\Omega)$.

Proof follows from the complete continuity of the nonlinear operator standing in parentheses in (11), as an operator from $\widetilde{W}_2^4(\Omega)$ into $\mathcal{L}_2(\Omega)$, which is established with the aid of embedding theorems.

3. Existence theorem.

By a generalized solution of problem (2)–(3) we shall mean a function $\psi(x)$ satisfying (10), i.e., any fixed point of the operator P. A generalized solution satisfies the boundary conditions (3) and, almost everywhere, equation (2). In proving the existence of a solution we shall use the Leray–Schauder principle ([1]).

Theorem 1. *There exists at least one solution of problem (2)–(3) for any function $f(x_1, x_2)$ belonging to the space $\mathcal{L}_2(\Omega)$.*

Proof. It is enough to show that the norms of all possible solutions $\psi^{(\lambda)}$ of the equations

$$\psi - \lambda P\psi = 0 \tag{12}$$

for $\lambda \in [0,1]$ are bounded in the aggregate by one and the same number. Suppose that $\|\psi^{(\lambda)}\|_{W_2^4}$ for all $\lambda \in [0,1]$ are not bounded in the aggregate. Then there exists a sequence $\lambda_1, \dots, \lambda_n, \dots$ from $[0,1]$, $\lambda_n \to \lambda_0$, for which the corresponding solutions $\psi_n = \psi(x, \lambda_n)$ of equations (12) have $\|\psi_n\|_{W_2^4} \to \infty$. Since ψ_n are solutions of equation (12), it follows from relation (5) that

$$\|\Delta\psi_n\|_{\mathcal{L}_2} \leq \frac{\gamma}{\alpha}\|f\|_{\mathcal{L}_2}. \tag{13}$$

From the embedding theorems and the results ([3]), using (13), we obtain

$$\|\partial\psi_n/\partial x_i\|_{\mathcal{L}_4} \leq C_2\|f\|_{\mathcal{L}_2}, \qquad \|\partial\psi/\partial x_i\|_{\mathcal{L}_2} \leq C_3\|f\|_{\mathcal{L}_2} \qquad (i = 1, 2). \tag{14}$$

Further, we have

$$\psi_n = \lambda_n P\psi_n.$$

Estimating the right-hand side in the norm of the space $W_2^4(\Omega)$, we obtain

$$\|\psi_n\|_{W_2^4} \leq C_4\|B\|_{\mathcal{L}_2 \to W_2^4}(\|\psi_n\|_{W_4^3} + 1), \tag{15}$$

where

$$C_4 = \frac{\|f\|_{\mathcal{L}_2}}{\alpha} \max\{2\beta C_2,\ C_3,\ 1\}.$$

Applying to (15) the multiplicative inequality ([4–6]) and relation (13), we obtain

$$\|\psi_n\|_{W_2^4} \le C_4\|B\|(C_5\|\psi_n\|_{W_2^4}^{7/8}+1),$$

where

$$C_5 = \left(\frac{\gamma^3}{\alpha}\|f\|_{\mathcal{L}_2}\right)^{1/7}.$$

Dividing both sides by the norm of ψ_n in $W_2^4(\Omega)$ and passing to the limit as $n\to\infty$, we arrive at a contradiction. The theorem is proved.

4. Uniqueness of the solution.

Theorem 2. *If the function $f(x)$ is such that*

$$\|f\|_{\mathcal{L}_2} < \alpha^2/6\gamma^3\beta, \tag{16}$$

then the solution of problem (2)–(3) is unique.

Proof. Let there exist two solutions $\psi_1,\ \psi_2$; then for $\psi=\psi_1-\psi_2$ we have the relation

$$\|\Delta\psi\|_{\mathcal{L}_2}^2 = \frac{\beta}{\alpha}[(A\psi_2,\psi_1)+(A\psi_1,\psi_2)]. \tag{17}$$

Further, we have

$$(A\psi_1,\psi_2)+(A\psi_2,\psi_1) = \int_\Omega \Delta\psi\left(\frac{\partial\psi}{\partial x_2}\frac{\partial\psi_2}{\partial x_1} - \frac{\partial\psi}{\partial x_1}\frac{\partial\psi_2}{\partial x_2}\right)dx. \tag{18}$$

Using inequalities (7), (13), and (18), we obtain

$$(A\psi_1,\psi_2)+(A\psi_2,\psi_1) \le \frac{6\nu^3}{a}\|\Delta\psi\|_{\mathcal{L}_2}^2\,\|f\|_{\mathcal{L}_2}. \tag{19}$$

From (17), (19), and (16) it follows that

$$\|\Delta\psi\|_{\mathcal{L}_2} < \|\Delta\psi\|_{\mathcal{L}_2}. \tag{20}$$

The latter is possible when $\Delta\psi=0$, and, by virtue of (3), we obtain that $\psi=0$.

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Note: Figure translations are in progress. See original paper for figures.