THE PRINCIPLE OF MAXIMUM STABILITY OF AVERAGED TURBULENT FLOWS

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Abstract

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M. A. GOLDSTIK

THE PRINCIPLE OF MAXIMUM STABILITY OF AVERAGED TURBULENT FLOWS

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At the present time, as is well known, there is no unified concept that would make it possible to calculate turbulent flows theoretically. The statistical theory of turbulence, despite certain successes, is apparently still far from this goal. It therefore seems tempting to formulate some qualitative heuristic principle that would make it possible, without recourse to experiment, to calculate the principal characteristics of a turbulent flow.

The first attempt of this kind belongs to Malkus (^1), who proposed that an averaged turbulent flow is a neutrally stable flow for which the rate of energy dissipation is maximal. In our opinion such a point of view is unacceptable, since it is difficult to admit that there exists some selected disturbance of the mean profile of a turbulent flow that would not decay, in contrast to disturbances of all other wavelengths. As experience shows, any disturbances introduced into an averaged turbulent flow, owing to the intensive mixing action of turbulent pulsations, decay, and quite rapidly. It is therefore not surprising that Malkus’ s theory did not withstand the numerical test carried out by the method of Lin by Reynolds and Tiederman (^2). These authors established that, first, the experimental turbulent profile is very stable and, second, the class of neutrally stable velocity profiles does not contain an element realizing the maximum rate of energy dissipation.

The aim of the present note is to formulate a new extremal hypothesis and to present the results of its numerical verification.

The proposed hypothesis consists in the assumption that an averaged turbulent flow possesses the property of maximum stability with respect to the most “dangerous” disturbance, i.e., one that decays more slowly than the others. One may imagine that an averaged turbulent flow is capable of assuming a number of stable states; however, taking into account the high dynamism of the turbulent system, the greatest probability of realization will belong to the state with maximum stability, if such a state exists.

If one uses the method of small disturbances and assumes that, in the first approximation, the tensor of turbulent stresses under disturbances of the mean velocity does not change (other possibilities are discussed below), then for the one-dimensional case, by standard transformations, it is not difficult to obtain the usual Orr–Sommerfeld equation:

$$\varphi^{\mathrm{IV}} - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha\,\mathrm{Re}[(u-c)(\varphi'' - \alpha^2\varphi) - u''\varphi]. \tag{1}$$

Here $\varphi(y)$ is the sought complex amplitude of the disturbed stream function, α^{-1} is the dimensionless wavelength of the superposed harmonic disturbance in the longitudinal direction, Re is the Reynolds number, and u is the dimensionless velocity of the averaged flow whose stability is being investigated.

If one considers the case of a plane channel and restricts the investigation to symmetric disturbances only, then for the function φ the following boundary conditions must be imposed

$$\varphi'(0) = \varphi'''(0) = 0, \tag{2}$$

$$\varphi(1) = \varphi'(1) = 0. \tag{3}$$

For a fixed profile u, the problem of its stability consists in determining the complex eigenvalue $C = X + iY$ having the maximum imaginary part. If $Y > 0$, the profile is unstable; for $Y < 0$ it is stable. The quantity Y depends on the parameter a. The most dangerous disturbance corresponds to that a which realizes the maximum of $Y(a)$, denoted below by Π:

$$\Pi = \max_a Y. \tag{4}$$

In the usual formulation of the stability problem, the function u is the solution of the Navier–Stokes equations and is regarded as given. In the proposed formulation the function u may be an arbitrary element of the space C_2 of twice continuously differentiable functions satisfying the boundary conditions (no slip) and the condition of constancy of the flow rate in the pipe:

$$\int_0^1 u\,dy = 1. \tag{5}$$

The formulated variational problem reduces to finding the expression

$$\inf_u \Pi = \inf_u \max_a Y \tag{6}$$

under the additional condition (5).

In this formulation the problem was posed by the author in Novosibirsk at the All-Union Thermophysical Seminar in 1963. At that time an attempt was made to test this hypothesis by means of a numerical experiment. However, it was not possible to overcome the computational difficulties. At the end of 1967, the author, together with V. A. Sapozhnikov, developed a new algorithm for solving equation (1), consisting of the following.

From the four linearly independent solutions of equations (1), choose two that satisfy conditions (2). Then, in order to satisfy conditions (3), it is necessary to require that the function vanish,

$$\psi = \varphi_1'/\varphi_1 - \varphi_2'/\varphi_2 \quad \text{for } y = 1, \tag{7}$$

so that the problem reduces to solving the equation $\psi(c) = 0$. The function $\psi(c)$ was determined by solving the transformed equation (1) by the Runge–Kutta method. The zeros of $\psi(c)$ were found by the secant method, modified for complex functions.

The main advantage of this method consists in the arbitrary choice of the integration step and the possibility of computations with controlled accuracy.

Since the purpose of this work was only to test the proposed hypothesis, as the set of trial functions u we restricted ourselves to considering the class of profiles used in work (^{2}).

$$\frac{du}{dy_1} = \mathrm{Re}\, B\, \frac{1 - y_1}{1 + E}; \qquad u(0) = 0; \tag{8}$$

$$E = \frac{1}{2}\left\{1 + \frac{1}{9}\chi^2 R^2 B \left[y_1(2 - y_1)(3 - 4y_1 + 2y_1^2)\times\right.\right.$$

$$\left.\left.\times\left(1 - \exp\left[-\frac{R\sqrt{B}}{A}\, y_1(2 - y_1)\right]\right)\right]^2\right\}^{1/2} - \frac{1}{2}, \tag{9}$$

where $y_1 = 1 - y$ is the distance from the channel wall; B is a parameter determined from relation (5); A is a parameter fixed in the present calculations ($A = 31$); $\varkappa$ is a variable parameter characterizing the profile. For $\varkappa = 0$ the velocity profile is a Poiseuille parabola; for $\varkappa = \infty$, $u \equiv 1$; finally, for $\varkappa \simeq 0.4$, relations (8)-(9) approximate Laufer' s experimental data on turbulent flow in a plane channel well.

The results of computations on the BESM-6 electronic computer for the value $\mathrm{Re} = 10^4$ are presented in Fig. 1, where the parameter $\varkappa$ is plotted along the abscissa and the function $\Pi(\varkappa)$ along the ordinate. As can be seen, the function $\Pi(\varkappa)$ has two branches, corresponding, for each fixed $\varkappa$, to the existence of two maxima of the function $Y(a)$, one of which lies in the region of short-wave disturbances (large a), while the other corresponds to long-wave disturbances

Fig. 1

Figure 1: Fig. 1

(small a). For $\varkappa < 0.526$ the most dangerous disturbances are short-wave; for $\varkappa > 0.526$, on the contrary, the role of the most dangerous passes to long-wave disturbances. The point $\varkappa = 0.526$ corresponds to the profile that is maximally stable with respect to the most dangerous disturbances: short-wave disturbances with $a \simeq 100$ and long-wave disturbances with $a \simeq 0.2$. It is not difficult to see that any other $\varkappa$ corresponds to less stable profiles.

Fig. 1

The difference in the shape of the profile corresponding to the value $\varkappa = 0.526$ from the experimental one is comparatively small; therefore it may be concluded that, in the first approximation, the proposed variational principle has withstood a numerical test.

In the present calculation it was assumed that the role of the Reynolds stresses reduces only to the formation of the mean velocity field, whose stability may be analyzed without taking into account the influence of these stresses, so that the latter are regarded as a certain field of body forces independent of the state of motion. Therefore, strictly speaking, the results obtained should be interpreted as a solution of the problem of finding external forces that maximally stabilize a laminar flow.

As the next step (approximation), one should take into account the direct stabilizing action of turbulent stresses. For this purpose one may, for example, introduce a variable "turbulent viscosity" and regard it as an independently varied function to be determined from the condition of maximum stability of the corresponding mean profile of the mean velocity.

Extending the range of Reynolds numbers, as well as implementing the indicated step, will apparently make it possible to clarify in greater detail the physical meaning of the proposed principle. Its definitive confirmation would open such interesting prospects as an explanation of the phenomenon of intermittency in pipes under transitional regimes, the nonuniqueness of certain flows, and the phenomenon of hydrodynamic hysteresis. It is also possible that this principle would prove useful in considering phenomena connected with plasma instability.

Institute of Thermophysics
Siberian Branch of the Academy of Sciences of the USSR

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CITED LITERATURE

1. W. V. R. Malkus, *J. Fluid Mech.*, **1**, 521 (1956).

2. W. C. Reynolds, W. G. Tiederman, *J. Fluid Mech.*, **27**, 253 (1967).

Note: Figure translations are in progress. See original paper for figures.