QUASILINEAR ELLIPTIC EQUATIONS AND THE CONSTRUCTION OF A HYPERSURFACE WITH PRESCRIBED MEAN CURVATURE

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Abstract

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QUASILINEAR ELLIPTIC EQUATIONS AND THE CONSTRUCTION OF A HYPERSURFACE WITH PRESCRIBED MEAN CURVATURE

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1. One of the most interesting problems of geometry leading to boundary-value problems for quasilinear elliptic equations is the problem of constructing a surface from its mean curvature.

We give the formulation of this problem. Let E^{n+1} be $(n+1)$-dimensional Euclidean space with Cartesian coordinate system $x_1, x_2, \dots, x_n; z$. We denote the plane $z = 0$ by E^n. Points of E^{n+1} will be denoted briefly by (x, z), and points of E^n by x. Let Ω be a bounded domain in E^n, homeomorphic to an n-dimensional closed ball, and let $\Gamma = \partial\Omega$. Suppose a function $H(x)$ is prescribed in Ω, and a continuous function $h(x)$ on Γ. The problem of constructing a hypersurface from its mean curvature is reduced to constructing in Ω a function $z(x)$ such that $z|_\Gamma = h(x)$ and the mean curvature of the surface Φ_z—the graph of the function $z(x, y)$—coincides with the function $H(x)$. Since for the mean curvature H of the surface Φ_z the formula

$$H = \sum_{i,k=1}^{n} G_{ik}(Dz) z_{ik} / n \left(1 + \sum_{i=1}^{n} z_i^2\right)^{3/2}, \tag{1}$$

holds, where

$$G_{ii} = 1 + \sum_{j \neq i} z_j^2, \quad G_{ik} = -z_i z_k, \quad z_i = \partial z / \partial x_i, \quad z_{ik} = \partial^2 z / \partial x_i \partial x_k,$$

the geometric problem formulated above is reduced to the solution of the following boundary-value problem:

$$\sum_{i,k=1}^{n} G_{ik}(Dz)z_{ik} = nH(x)\left(1+\sum_{i=1}^{n} z_i^2\right)^{3/2}, \tag{2}$$

$$z|_{\Gamma} = h(x). \tag{3}$$

Denote by P the n-dimensional Euclidean space with Cartesian coordinates $p_1, p_2, \dots, p_n$. A point (vector) of this space will be denoted by p. As usual, $|p|$ denotes the length of the vector p, i.e.

$$|p| = \left(\sum_{i=1}^{n} p_i^2\right)^{1/2}.$$

It is easily verified that for all $p \in P$ and all real numbers $\xi_1, \xi_2, \dots, \xi_n$ the exact inequalities

$$\sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,k=1}^{n} G_{ik}(p)\xi_i\xi_k \leq (1+|p|^2)\sum_{i=1}^{n} \xi_i^2 \tag{4}$$

hold.

It follows from inequalities (4) that equation (2) is elliptic, but not uniformly elliptic. Furthermore, the right-hand side of equation (2), with respect to $\sum_{i=1}^{n} z_i^2$, as $\sum_{i=1}^{n} z_i^2 \to +\infty$, has a higher order of growth than the expres-

$\sum G_{ik}(Dz)\xi_i\xi_j$, $\sum G_{ik}z_iz_k$. These circumstances exclude equation (2) from the classes of quasilinear elliptic equations for which, in ([2],[3]), various boundary-value problems have been thoroughly studied. Let us note that the above-indicated properties of the functions $G_{ik}(p)$ lead to adjoining necessary and sufficient conditions for solvability of the boundary-value problem (2)–(3), which are expressed by inequalities between the geometric characteristics of the domain Ω and the properties of the function $H(x)$ (see §§ 3, 4, 5, 6 of the present paper). These circumstances lead to the separation of a certain rather broad class of quasilinear elliptic equations, including both the equations studied in the above-cited works and equation (2). We shall denote this class of equations by H.

2. On the class of quasilinear elliptic equations H. Let, in the cylinder $\Omega+\Gamma\times J\times P$ with Cartesian coordinates $x_1, x_2, \dots, x_n$; z; $p_1, \dots, p_n$, where $J = (-\infty, +\infty)$, the functions $a_{ik}(x,z,p)$ $(i,k = 1,\dots,n)$, $b(x,z,p)$ satisfy the following conditions: 1) for arbitrary $\xi_1, \xi_2, \dots, \xi_n$ in $\Omega+\Gamma\times J\times P$ the inequality

$$\sum_{i,k=1}^{n} a_{ik}(x,z,p)\xi_i\xi_k > 0;$$

2) for all $x \in \Omega + \Gamma,\ z \in J,\ p \in P$ the inequalities

$$-\varphi_{-}(x)/R_{-}(x) \leq b(x, z, p)\,[\det \|a_{ik}(x, z, p)\|]^{-1/n} \leq \varphi_{+}(x)/R_{+}(p), \quad (5)$$

hold, where $R_{\pm}(p) > 0$ are locally summable to degree n in P, and $\varphi_{\pm}(x) \geq 0$ belong to $L_n(\Omega)$.

In this paper we shall restrict ourselves to the case where the function $b[\det \|a_{ik}\|]^{-1/n}$ satisfies inequalities (5). However, with minor changes the results of the present paper carry over to the case where the function $b[\det \|a_{ik}\|]^{-1/n}$ increases with respect to z for fixed values of the remaining variables.

Quasilinear equations of the form

$$\sum_{i,k=1}^{n} a_{ik}(x, z, Dz) z_{ik} = b(x, z, Dz) \quad (6)$$

will be called **equations of class** H if the functions a_{ik} and b satisfy conditions 1), 2).

3. Necessary conditions for solvability of the first boundary-value problem. Let $z(x) \in C^2(\Omega)$ be a solution of the boundary-value problem

$$\sum_{i,k=1}^{n} a_{ik}(x, z, Dz) z_{ik} = b(x, z, Dz), \qquad z\big|_{\Gamma} = h(x), \quad (7)$$

where in (7) the equation belongs to the class H, and $h(x) \in C(\Gamma)$. Denote by $u(x)$ and $v(x)$ the convex functions stretched respectively from below and from above over the function $z(x)$. (By this term, according to ([1]), one means

$$u(x) = \sup_{W^{+}}\{w(x)\},$$

where W^{+} is the totality of convex functions satisfying in Ω the inequality $w(x) \leq z(x)$ and directed by convexity downward. The function $v(x)$ is defined analogously.) Let M_u and M_v be, respectively, the sets of interior points of Ω at which $u(x) = z(x)$ and $v(x) = z(x)$. Then the following is valid.

Theorem 1. For any solution $z(x) \in C^2(\Omega)$ of the boundary-value problem (7), the relations

$$\int_{\nu_z(M_u)} R_{+}^{n}(p)\,dp \leq \frac{1}{n^n}\int_{M_u} [\varphi_{+}(x)]^n\,dx, \qquad \int_{\nu_z(M_v)} R_{-}^{n}(p)\,dp \leq \frac{1}{n^n}\int_{M_v} [\varphi_{-}(x)]^n\,dx, \quad (8)$$

necessarily hold, where ν_z is the mapping of Ω into P given by the formulas $p_1 = \partial z/\partial x_1, \dots, p_n = \partial z/\partial x_n$.

4. Two-sided estimates of the solution of the boundary-value problem (7).

Let $f(p) > 0$ be a locally summable function in the space P. Put

$$g(\rho) = \int_{|p|\le\rho} f(p)\,dp, \qquad A(f) = \int_P f(p)\,dp.$$

Since $g(\rho)$ is a strictly increasing continuous function of ρ on $[0, +\infty]$, $g(\rho)$ has an inverse function $\rho = F(f, \tau)$, strictly increasing and continuous on the interval $[0, A(f))$, and

$$\lim_{\tau\to A(f)} F(f, \tau) = +\infty.$$

Theorem 2. Let $z(x) \in C^2(\Omega)$ be a solution of the boundary-value problem (7). Then, if for the functions $\varphi_+(x)$, $\varphi_-(x)$, $R_+(p)$, $R_-(p)$ the inequalities

$$\omega_\pm = \frac{1}{n^n}\int_\Omega [\varphi_\pm(x)]^n\,dx < A(R_\pm^n), \tag{9}$$

hold, then for $z(x)$ in $\Omega + \Gamma$ the estimates

$$h_2 - F(R_+^n, \omega_+)d \le z(x) \le h_1 + F(R_-^n, \omega_-)d, \tag{10}$$

are valid, where

$$h_1 = \sup_\Gamma h(x), \qquad h_2 = \inf_\Gamma h(x), \qquad d$$

is the diameter of Ω.

For the problem with mean curvature (equation (2)) these conditions and estimates take the form

$$\omega_\pm = \int_\Omega H_\pm^n(x)\,dx < \sigma_n^*, \tag{11}$$

$$h_2 - \left[\frac{\omega_+^{2/n}}{\sigma_n^{2/n} - \omega_+^{2/n}}\right]^{1/2} d \le z(x) \le h_1 + \left[\frac{\omega_-^{2/n}}{\sigma_n^{2/n} - \omega_-^{2/n}}\right]^{1/2} d, \tag{12}$$

where σ_n is the volume of the n-dimensional ball of radius one. If Ω is a ball, $H(x) = H_0 = \text{const} > 0$ in Ω and $z|_\Gamma = 0$, then inequality (11) takes the form

$$rH_0 < 1,$$

where r is the radius of the ball Ω. It is easy to show that inequality (11) in this special case is a necessary condition for solvability of the boundary-value problem (7) in the class of functions $C^2(\Omega + \Gamma)$.

5. Estimates of the normal derivative on Γ for a solution of problem (7).

Let Ω be a domain in E^n, bounded by a continuously differentiable closed surface Γ. Let S be an $(n-1)$-dimensional surface in E^{n+1}, constructed from the boundary condition

$$x|_\Gamma = h(x) \in C^1(\Gamma).$$

Let X be any point of S, and Q a hyperplane in E^{n+1} passing through the tangent plane T to S at the point X and leaving S above itself. Let

$$\tilde{z} = \sum_{i=1}^{n} a_i x_i + b$$

be the equation of Q. The number

$$M_+(S) = \sup_S \left\{ \inf \sum_{i=1}^{n} a_i^2 \right\},$$

where the exact lower bound is taken over all planes Q passing through T and leaving S above themselves, will be called the lower bending of the surface S. Similarly one introduces the upper bending $M_-(S)$. If S and $\Gamma = \partial\Omega$ belong to C^2, and at each point $x \in \Gamma$ all normal curvatures of this surface are bounded below by one and the same number $\chi_0 > 0$, then the numbers $M_\pm(S)$ are finite. Here and below we shall assume that the functions $R_\pm(p)$, occurring in inequalities (5), depend on the length of the vector p and are continuous on P. Then the functions

$$N_\pm(|p|, R_\pm^n(|p|)) = \inf R_\pm^n(|p|),$$

where the exact lower bounds are taken respectively in balls of radii $\sqrt{M_+}$ and $\sqrt{M_-}$ with center at the point p, are also continuous functions of $|p|$.

* By $H_\pm(x) \geq 0$ are denoted, respectively, the positive and negative parts of the function $H(x)$.

Theorem 3. If $z(x) \in C^2(\Omega + \Gamma)$ is a solution of the boundary-value problem (7) and the surfaces S and $\Gamma \in C^{2,\beta}$, and if the normal curvatures at each point of Γ are bounded below by one and the same number χ_0, then, provided the inequalities

$$\psi_\pm \equiv \frac{1}{n^n \chi_0^n} \sup_\Omega [\varphi_\pm(x)]^n < \int_{\dot{P}} N_\pm(|p|, R_\pm^n(|p|))\, dp \tag{13}$$

hold, the following estimates hold on Γ:

$$\sqrt{M_+(S)} + F(N_+, \psi_+) \geq \left.\frac{\partial z}{\partial n}\right|_\Gamma \geq -\sqrt{M_-(S)} - F(N_-, \psi_-). \tag{14}$$

6. The problem of constructing a hypersurface with prescribed mean curvature. For lack of space, in this article we shall confine ourselves only to formulations of existence theorems for the solution of the first boundary-value problem for equation (2), i.e. the case of constructing a hypersurface with prescribed mean curvature.

Theorem 4. Let the domain Ω and the function $h(x)$ prescribed on its boundary belong to $C^{m,\beta}$ $(m \geq 3,\ 0 < \beta < 1)$, and suppose, moreover, that the normal curvatures at the boundary points of Ω are bounded below by a constant $\chi_0 > 0$. Suppose further that $H(x) \in C^{m-2,\beta}$ and that the conditions

$$H_{\pm} < \chi_0 T_n(\sqrt{M_{\pm}(S)}), \tag{15}$$

are satisfied, where

$$H_{+} = \max\{\sup_{\Omega} H(x), 0\}, \qquad H_{-} = \max\{-\inf_{\Omega} H(x), 0\};$$

$$T_n(Q) = \left[n \int_0^{+\infty} p^{n-1} \big[1 + (p+Q)^2\big]^{-(n+2)/2} \, dp \right]^{1/n}.$$

Then the boundary-value problem of constructing a surface with prescribed mean curvature equal to $H(x)$ in Ω has in $C^2(\Omega)$ a unique solution which, moreover, belongs to the space $C^{m,\delta}(\Omega + \Gamma)$ $(0 < \delta < \beta)$.

Condition (15) is a necessary solvability condition for the problem under consideration if Ω is a ball, $H(x) = \text{const}$, and $z|_{\partial\Omega} = \text{const}$ (in this case $M_{\pm}(S) = 0$). If one imposes on the boundary of the domain Ω restrictions that are, in a certain sense, stronger, then one can obtain conditions under which the problem under consideration is solvable for arbitrary regular boundary conditions.

Theorem 5. Let the domain Ω and the function $h(x)$ prescribed on its boundary belong to $C^{m,\beta}$ $(m \geq 3,\ 0 < \beta < 1)$. Suppose further that $H(x) \in C^{m-2,\beta}$ and that at all points of $\Gamma = \partial\Omega$ the inequality

$$L \geq \frac{n}{n-1} \sup_{\Omega+\Gamma} |H(x)|$$

holds, where L is the mean curvature of Γ in the direction of the inward normal to Γ. Then the boundary-value problem of constructing a surface whose mean curvature coincides with $H(x)$ and whose boundary is $z|_{\partial\Omega} = h$ has in C^2 a unique solution which, moreover, belongs to $C^{m,\delta}(\Omega + \Gamma)$ $(0 < \delta < \beta)$.

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