EXTENSION OF THE QUASIANALYTIC CLASSES OF DENJOY–CARLEMAN

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Abstract

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EXTENSION OF THE QUASIANALYTIC CLASSES OF DENJOY–CARLEMAN

The solution of J. Hadamard' s problem on the quasianalyticity of the class $C\{M_n\}$ of functions $\varphi(x)$, infinitely differentiable on a certain interval $I=(a,b)$, subject to conditions of the form

$$\left|\varphi^{(n)}(x)\right| \leq A \cdot B^n M_n \qquad (n=1,2,\ldots), \tag{1}$$

where $A=A(\varphi)>0$ and $B=B(\varphi)>0$ are constants, was given partially by Denjoy (^{1}) and in final form by Carleman (^{2}).

In A. Ostrowski' s formulation (^{3}), Carleman' s theorem states:

In order that the class $C\{M_n\}$ be quasianalytic, i.e., in order that for every function $\varphi(x) \in C\{M_n\}$, from the equalities $\varphi^{(n)}(x_0)=0,\ x_0 \in I\ (n=0,1,2,\ldots)$ there should follow the identity $\varphi(x) \equiv 0,\ x \in I$, it is necessary and sufficient that the integral

$$\int_1^\infty \frac{\log T(r)}{r^2}\, dr = +\infty, \qquad T(r)=\sup_{n\geq 1} \frac{r^n}{M_n}. \tag{2}$$

diverge.

In connection with this theorem it is natural to pose the following question:

If the integral (2) converges and, therefore, the class $C\{M_n\}$ is non-quasianalytic on $[0,+\infty]$ or on $[0,l]$, then what data, instead of the sequence of values $\varphi^{(n)}(0)$ $(n=0,1,2,\ldots)$, determine the functions of this class uniquely?

In the present note we introduce the concept of α-quasianalyticity, which includes, in particular, the concept of classical quasianalyticity, and give formulations of certain basic theorems that provide a complete solution of the problem posed.

1°. Let the function $\varphi(x)$ be defined and measurable on $(0,+\infty)$. Then on $(0,+\infty)$ one may introduce for consideration the following operators:

$$D_\infty^{-\alpha}\varphi(x) \equiv \frac{1}{\Gamma(\alpha)}\int_x^{+\infty}(t-x)^{\alpha-1}\varphi(t)\,dt \quad (0<\alpha<+\infty),$$

$$D_\infty^{\alpha}\varphi(x) \equiv \frac{d}{dx}D^{-(1-\alpha)}\varphi(x) \quad (0<\alpha\le 1), \tag{3}$$

known under the name of the integral and, respectively, derivative in Weyl' s sense of order α. It is then natural to identify the integral or derivative of zero order with the function itself, i.e., to set

$$D_\infty^{-0}\varphi(x) = D_\infty^{0}\varphi(x) = \varphi(x). \tag{3'}$$

Consider the set $C_\alpha^{(\infty)}$ $(0\le\alpha<1)$ of functions $\varphi(x)$, infinitely differentiable on $[0,+\infty)$, subject to the conditions

$$\sup_{0\le x<+\infty}\left|(1+x^{\alpha m})\varphi^{(n)}(x)\right| < +\infty \qquad (n,m=0,1,2,...). \tag{4}$$

Further, under the assumption that $\varphi(x)\in C_\alpha^{\infty}$ $(0\le\alpha<1)$, putting $1/\rho = 1-\alpha$ $(\rho\ge 1)$, we consider the operators of sequential differenti-

of orders n/ρ $(n=0,1,2,...)$ in the Weyl sense

$$D_\infty^{0/\rho}\varphi(x) \equiv \varphi(x), \qquad D_\infty^{1/\rho}\varphi(x) \equiv \frac{d}{dx}D_\infty^{-\alpha}\varphi(x),$$

$$D_\infty^{n/\rho}\varphi(x) \equiv D_\infty^{1/\rho}D_\infty^{(n-1)/\rho}\varphi(x) \quad (n=2,3,...). \tag{5}$$

Finally, for an arbitrary sequence of positive numbers $\{M_n\}_1^\infty$ we introduce the following two classes of infinitely differentiable functions:

Class $C_\alpha^*\{[0,+\infty);M_n\}$—the set of functions in $C_\alpha^{(\infty)}$ satisfying the conditions

$$\sup_{0\le x<+\infty}\left|D_\infty^{n/\rho}\varphi(x)\right| \le A\cdot B_n^{nM} \quad (n=1,2,...). \tag{6}$$

Class $C_\alpha\{[0,+\infty);M_n\}$—the set of functions $\varphi(x)$ in $C_\alpha^{(\infty)}$ satisfying the conditions

$$\sup_{0\le x<+\infty}\left|(1+\alpha x^2)\varphi^{(n)}(x)\right| \le A\cdot B_n^{nM} \quad (n=1,2,3,...), \tag{7}$$

where $A=A(\varphi)>0$ and $B=B(\varphi)>0$ are constants depending, generally speaking, on the particular function of the given class.

For both of these classes we pose a question analogous to Adamar' s problem and reducing to that same problem when the value of the parameter $\alpha = 0$.

What must the sequence of numbers $\{M_n\}_1^\infty$ be like in order that for every function $\varphi(x)$ from the corresponding class $C_\alpha^*\{[0,+\infty); M_n\}$ or $C_\alpha\{[0,+\infty); M_n\}$, from the equalities

$$D_\infty^{n/\rho}\varphi(0) = \frac{1}{\Gamma(n\alpha)}\int_0^{+\infty} x^{n\alpha-1}\varphi^{(n)}(x)\,dx = 0 \quad (n = 0, 1, 2, ...) \tag{8}$$

there would follow the identity $\varphi(x) \equiv 0,\ 0 \leq x < +\infty$?

Classes of this kind $C_\alpha^*\{[0,+\infty); M_n\}$ or $C_\alpha\{[0,+\infty); M_n\}$ will henceforth be called α-quasianalytic. Let us add that the 0-quasianalytic classes $C_0^*\{[0,+\infty); M_n\}$ and $C_0\{[0,+\infty); M_n\}$ are nothing other than the quasianalytic class $C\{M_n\}$ on $[0,+\infty)$ in the usual classical sense.

The following basic theorems have been established, giving an answer to the question of α-quasianalyticity for both of the classes introduced above, in terms of the same function

$$T(r) = \sup_{n\geq 1} r^n/M_n. \tag{9}$$

Theorem 1. In order that the class $C_\alpha^*\{[0,+\infty); M_n\}$ $(0 \leq \alpha < 1)$ be α-quasianalytic, it is necessary and sufficient that the condition

$$\int_1^{+\infty} \frac{\log T(r)}{r^{1+1/(1+\alpha)}}\,dr = +\infty \tag{10}$$

be satisfied.

Theorem 2. In order that the class $C_\alpha\{[0,+\infty); M_n\}$ $(0 \leq \alpha < 1)$ be α-quasianalytic, it is necessary and sufficient that the condition

$$\int_1^{+\infty} \frac{\log T(r)}{r^{1+(1-\alpha)/(1+\alpha)}}\,dr = +\infty. \tag{11}$$

Let us note that each of these theorems, in the case when the value of the parameter $\alpha = 0$, reduces to the classical Denjoy-Carleman theorem for the half-line $[0,+\infty)$.

We note that elementary estimates show that if

$$M_n = \left(n^{(1+\alpha)/(1-\alpha)} \log n \ldots \log_p n\right)^n, \qquad n \geq N_p, \tag{12}$$

where $p \geq 1$ is any integer, then condition (11) is satisfied.

Thus, in this case the α-quasianalytic class $C_\alpha\{[0,+\infty);M_n\}$ $(0<\alpha<1)$ is essentially broader than the ordinary quasianalytic class $C_0^\infty\{[0,+\infty);M_n\}\equiv C\{M_n\}$, since in the first case the successive derivatives of functions of the class may grow considerably faster $((1+\alpha)/(1-\alpha)>1$, for $0<\alpha<1)$ than is permissible for classes quasianalytic in the ordinary sense, as is seen, for example, from the original result of Denjoy, when

$$M_n=(n\log n\dots\log_p n)^n,\qquad n\geq N_p. \tag{12'}$$

Let us also note that the main difficulty in proving these theorems falls on theorem 1, by means of which theorem 2 is then established comparatively easily.

As for the proof of theorem 1, like the original proof of the Denjoy–Carleman theorem, it is also carried out by the method of reducing the problem of α-quasianalyticity to the known Watson problem, but now for an angular domain of opening $\pi(1+\alpha)$, and it relies essentially also on the existence of a nonzero analytic function that decreases extremely rapidly in this angle.

However here, as in the analogous problem for the finite interval $[0,\sigma]$, which will be discussed below, such a reduction of the problem can be effected only with the aid of the apparatus of integral transformations and representations with the Mittag–Leffler kernel

$$E_\rho(z;\mu)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu+k/\rho)}, \tag{13}$$

developed in the investigations of the author of article (^{4}), with the simultaneous use of differentiation operators D $(n=0,1,2,\dots)$ in the sense of Weyl.

In doing so we have to rely essentially on the following two auxiliary theorems on integral representation, close in spirit to certain earlier results of the author concerning generalizations of the Wiener–Paley theorem (^{4}).

Theorem 3. Let the function $f(z)$ be analytic inside and continuous in the closed domain

$$\overline{\Delta}_\rho^*=\{z;\ \pi/2\rho\leq|\arg z|\leq\pi,\ 0<|z|<+\infty\}\qquad(\rho>1/2), \tag{14}$$

and, in a neighborhood of the point $z=\infty$,

$$\max_{\pi/2\rho\leq|\varphi|\leq\pi}\{|f(re^{i\varphi})|\}=O(r^{-\omega})\qquad(\omega>1). \tag{15}$$

Then the representation

$$f(z) = \int_0^{+\infty} E_\rho(zt^{1/\rho};\, 1/\rho)t^{1/\rho-1}\varphi(t)\,dt, \qquad z \in \Delta_\rho^*, \tag{16}$$

holds, where

$$\varphi(t) = \frac{1}{2\pi i}\int_{L_\rho} e^{-t\xi^\rho} f(\xi)\,d\xi, \qquad t \in [0, +\infty), \tag{17}$$

and L_ρ is the boundary of the domain Δ_ρ^*, traversed in the positive direction.

Theorem 4. If $f(z)$ is an entire function of order $\rho > 1/2$ and type σ $(0 < \sigma < +\infty)$, satisfying condition (15) of theorem 3 for

$\omega > \max\{1, 1/\rho\}$, then the representation

$$f(z) = \int_0^{\sigma} E_\rho(zt^{1/\rho};\, 1/\rho)t^{1/\rho-1}\varphi(t)\,dt, \tag{18}$$

holds, where

$$\frac{1}{2\pi i}\int_{L_\rho} e^{-t\zeta^\rho} f(\zeta)\,d\zeta = \begin{cases} \varphi(t), & t \in [0, \sigma], \\ 0, & t \in [\sigma, +\infty) \end{cases} \tag{19}$$

In addition, in order to reduce the problem of α-quasianalyticity to Watson' s problem, we also rely on the properties of the functions $e^{-x\lambda^\rho}$ and $E_\rho(\lambda x^{1/\rho};\, 1/\rho)x^{1/\rho-1}$, which are solutions of certain special problems of the Cauchy type for fractional-order differential operators.

2°. The concept of α-quasianalyticity can also be introduced for classes of functions infinitely differentiable on a finite interval.

Let the function $\varphi(x)$ be defined and measurable on $[0, l]$ $(0 < l < +\infty)$. Then on $(0, l)$ one may also consider the following operators

$$D_l^{-\alpha}\varphi(x) \equiv \frac{1}{\Gamma(\alpha)}\int_x^l (x-t)^{\alpha-1}\varphi(t)\,dt \quad (0 < \alpha < +\infty),$$

$$D_l^{\alpha}\varphi(x) \equiv \frac{d}{dx}D_l^{-(1-\alpha)}\varphi(x) \quad (0 < \alpha \le 1), \tag{20}$$

known as the integral and, respectively, the derivative in the Riemann-Liouville sense of order α with terminal point at $x = l$, setting again

$$D_l^{-0}\varphi(x) = D_l^{0}\varphi(x) = \varphi(x). \tag{20'}$$

Further, for a fixed value of the parameter α $(0 \leq \alpha < 1)$, set again $1/\rho = 1 - \alpha$ $(\rho \geq 1)$, and on the interval $[0, l]$ consider the operators

$$D_l^0 \varphi(x) \equiv \varphi(x), \qquad D_l^{1/\rho} \varphi(x) \equiv \frac{d}{dx} D_l^{-\alpha} \varphi(x),$$

$$D_l^{n/\rho} \varphi(x) \equiv D_l^{1/\rho} D_l^{(n-1)/\rho} \varphi(x) \quad (n = 2, 3, \ldots). \tag{21}$$

We now define the following two classes of infinitely differentiable functions on $[0, l]$ $(0 < l < +\infty)$, associated with the given sequence of positive numbers $\{M_n\}_1^\infty$.

The class $C_\alpha^*\{[0, l]; M_n\}$ $(0 \leq \alpha < 1)$ consists of functions $\varphi(x)$ possessing on $[0, l]$ all successive derivatives in the Riemann-Liouville sense $D_l^{n/\rho} \varphi(x)$ $(n = 0, 1, 2, 3, \ldots)$, continuous on $[0, l]$, for which

$$\left| D_l^{n/\rho} \varphi(x) \right| \leq A \cdot B^n M_n; \qquad x \in [0, l] \quad (n = 1, 2, 3, \ldots) \tag{22}$$

and the class $C\{[0, l]; M_n\}$ consists of functions $\varphi(x)$, infinitely differentiable on $[0, l]$, for which

$$\left| \varphi^{(n)}(x) \right| \leq A \cdot B^n M_n \quad (n = 1, 2, \ldots). \tag{23}$$

Each of these classes will be called α-quasianalytic if, for any function $\varphi(x)$ from the corresponding class, the equalities

$$\varphi^{(n)}(l) = D_l^{n/\rho} \varphi(0) = \frac{1}{\Gamma(n\alpha)} \int_0^l x^{n\alpha - 1} \varphi(x) \, dx = 0 \quad (n = 0, 1, 2, \ldots) \tag{24}$$

imply that $\varphi(x) \equiv 0, \ x \in [0, l]$.

Theorems on the α-quasianalyticity of these classes have been established; we do not give their formulations here, since in essence they are the same as in the corresponding Theorems 1 and 2.

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Note: Figure translations are in progress. See original paper for figures.

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