ON NILPOTENT \(J\)-SELF-ADJOINT OPERATORS

MATHEMATICS

1968

SovietRxiv

Abstract

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UDC 513.88:517.948.35

MATHEMATICS

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ON NILPOTENT J-SELF-ADJOINT OPERATORS

(Presented by Academician I. N. Vekua on 11 III 1968)

Let $\mathfrak{H}$ be a J-space. This is the name for a Hilbert space in which, along with the usual scalar product $[x, y]$, an indefinite scalar product $(x, y) = [Jx, y]$ is given, where $J = P_+ - P_-$, P_+ and P_- are orthoprojectors, $P_+ + P_- = I$. By $\mathfrak{M}_+$ ($\mathfrak{M}_-$) we shall denote the totality of all maximal subspaces in $\mathfrak{P}_+ = \{x \in \mathfrak{H} : (x, x) \geq 0\}$ ($\mathfrak{P}_- = \{x \in \mathfrak{H} : (x, x) \leq 0\}$). Put $\mathfrak{P}_0 = \{x \in \mathfrak{H} : (x, x) = 0\}$.

The notions of a J-self-adjoint and a J-unitary operator in $\mathfrak{H}$ are introduced, with respect to the form (x, y), in the usual way. Linear operators whose spectrum is concentrated only at zero occupy a special place in the spectral theory of operators ([1,2]). Such operators are called nilpotent.

In contrast to the usual scalar product $[x, y]$, with respect to which there exists no nonzero nilpotent self-adjoint, or even normal, operator, there do exist nonzero J-self-adjoint nilpotent operators.

Let us give the simplest example. Introduce in the complex Euclidean space E_n of dimension $n = 2k$ the quadratic form

$$(x, x) = \xi_1\bar{\xi}_{2k} + \xi_2\bar{\xi}_{2k-1} + \dots + \xi_{2k}\bar{\xi}_1, \tag{1}$$

It is easy to verify that the form (1) contains k negative squares, so that the space E_n, furnished with the form (1) and the scalar product

$$[x, y] = \sum_{i=1}^{2k} \xi_i\bar{\eta}_i, \tag{2}$$

where $x = (\xi_1, \xi_2, \dots, \xi_{2k})$, $y = (\eta_1, \eta_2, \dots, \eta_{2k}) \in E_n$, becomes a finite-dimensional space with an indefinite metric. We shall denote such a space by Π_k^{2k}.

Now define an operator A in the space Π_k^{2k} by the following square matrix of order $2k + 1$

$$A = \begin{pmatrix} 0 & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,2k} \\ 0 & 0 & a_{2,k+2} & \cdots & a_{2,2k} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & a_{k,2k} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{3}$$

The operator A is nilpotent.

If now the matrix is self-adjoint with respect to the second main diagonal, then it is verified directly that, with respect to the form (1),

$$(Ax, y) = (x, Ay). \tag{4}$$

Let now A be a Volterra operator in $\mathfrak{H}$.

Consider in the space $\mathfrak{H}_0 = \mathfrak{H} \oplus \mathfrak{H}$ the operators H and J, whose matrix representation has the form

$$H = \begin{pmatrix} 0 & (A^*A)^{1/2} \\ -(A^*A)^{1/2} & (A + A^*) \end{pmatrix}, \qquad J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

It is easy to prove that $\sigma(H) = 0$. On the other hand,

$$(JH)^* = JH.$$

Denote by Ω the totality of all nilpotent J-self-adjoint operators in the space $\mathfrak{H}$.

Theorem 1. *If an operator $A \neq 0$ from Ω has an invariant subspace $\mathcal{L}$ from $\mathfrak{M}_+$ or $\mathfrak{M}_-$, then the subspace $\mathcal{L}$ necessarily contains a J-neutral subspace $\mathcal{L}_0$.*

Proof. Suppose that $\mathcal{L}$ is a J-definite invariant subspace of the operator A. For definiteness let $\mathcal{L} \in \mathfrak{M}_+$. Then the J-orthogonal complement $\mathcal{F}$ to the J-positive subspace $\mathcal{L}$ in $\mathfrak{H}$ will be a J-negative maximal invariant subspace of the operator A.

It is known (^{3}) that the direct J-orthogonal sum $\hat{\mathfrak{H}} = \mathcal{L} \oplus \mathcal{F}$ is dense in the norm $\|x\| = [x, x]^{1/2}$ in $\mathfrak{H}$. On the other hand, with respect to the definite scalar product

$$[x, y]_1 = (x_+, y_+) - (x_-, y_-), \qquad x_+, y_+ \in \mathcal{L}, \quad x_-, y_- \in \mathcal{F},$$

the restriction $\hat{A}$ of the operator A to the subspace $\hat{\mathfrak{H}}_1$ becomes a self-adjoint operator. It is easy to see that the operator $\hat{A}$ is equal to zero only together with the operator A. From the self-adjointness of $\hat{A}$ and $A \in \Omega$ it follows that $\hat{A} = 0$, which contradicts the fact that $A \neq 0$. The theorem is proved.

Corollary 1. *If an operator $A \in \Omega$ has an invariant subspace $\mathcal{L}$ from $\mathfrak{M}_+$ or $\mathfrak{M}_-$, then it has respectively the form*

$$\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_+ \quad \text{or} \quad \mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_-,$$

where $\mathcal{L}_0 \neq \{0\}$, and $\mathcal{L}_+$ and $\mathcal{L}_-$ are J-definite subspaces obtained by decomposing $\mathcal{L}$ with the aid of some skew-related (^{4}) *subspace $\mathcal{F}_0$ from $\mathfrak{H}$ with $\mathcal{L}_0$.*

Corollary 2. *Invariant subspaces from $\mathfrak{M}_+$ and $\mathfrak{M}_-$ of a Volterra J-self-adjoint operator A have the form indicated in Corollary 1.*

Let us note that the subspace $\mathcal{L}_0$ in Corollary 1 is invariant with respect to the operator A, since it is the intersection of the subspace $\mathcal{L}$ and its J-orthogonal complement, and both these subspaces are invariant with respect to the operator A.

Let us see how the operator A acts on the invariant subspace $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_+$ or $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_-$ for $\mathcal{L}_\pm \neq \{0\}$, when $\mathcal{L}_0$ is a maximal J-neutral invariant subspace of the operator A in $\mathcal{L}$. Let $\mathcal{L}_0^\perp$ be the J-orthogonal complement to $\mathcal{L}_0$ in $\mathfrak{H}$. Obviously, $\mathcal{L}_0 \subset \mathcal{L}_0^\perp$, since $\mathcal{L}_0 \subset \mathfrak{P}_0$. Let $\mathcal{F}_0 = J\mathcal{L}_0$. Put $\mathcal{E}_0 = \mathcal{L}_0 + \mathcal{F}_0$, and let $\mathcal{E}_0^\perp$ be the J-orthogonal complement to $\mathcal{E}_0$ in $\mathfrak{H}$. It is easy to show that $\mathcal{E}_0^\perp$ is a J-space, and therefore the factor space $\mathfrak{H}_1 = \mathcal{L}_0^\perp / \mathcal{L}_0$ is also a J-space. Let A_1 be the restriction of the operator A to $\mathfrak{H}_1$. Obviously, A_1 is a nilpotent J-self-adjoint operator. By assumption (and if A is Volterra, then by the theorem from (^{5})) there exists a J-nonnegative maximal invariant subspace $\mathcal{L}_1$ of the operator A_1 in $\mathfrak{H}_1$. Since $\mathcal{L}_0$ is a maximal J-neutral invariant ⋯

subspace of the operator A in $\mathcal{L}$, then $\mathcal{L}_1$ is J-positive, and therefore, as in Theorem 1, there exists a definite scalar product with respect to which the operator A_1 becomes self-adjoint. Thus, $A_1 = 0$, and therefore the operator A maps $\mathcal{L}_0^\perp$ into $\mathcal{L}_0$. Let $\mathcal{L}_2$ be the J-orthogonal complement in $\mathfrak{H}_1$ to the subspace $\mathcal{L}_1$. Putting now $\mathcal{L}_+ = \mathcal{L}_1$ and $\mathcal{L}_- = \mathcal{L}_2$, we obtain

$$A\mathcal{L} = A(\mathcal{L}_0 \oplus \mathcal{L}_+) = A\mathcal{L}_0 \oplus A\mathcal{L}_+ \subseteq \mathcal{L}_0. \tag{5}$$

and an analogous relation is obtained for $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_-$.

Consider the class Ω in the space Π_k.

Theorem 2. *Every operator A from Ω in the Pontryagin space Π_k is finite-dimensional, and its dimension does not exceed $2k$, where k is the finite rank of indefiniteness of the space Π_k.*

Proof. Let $\mathcal{L}_0$ be a maximal J-neutral invariant subspace of the operator A. If $A \neq 0$, then, by Theorem 1, $\mathcal{L}_0 \neq \{0\}$. On the other hand (4), $\dim \mathcal{L} \leq k$. Let $\mathcal{F}_0 = J\mathcal{L}_0$, $\mathcal{E}_0 = \mathcal{F}_0 \dotplus \mathcal{L}_0$. Then (4)

$$\Pi_k = \mathcal{F}_0 \dotplus \mathcal{L}_0 \oplus \mathcal{E}_0^\perp, \tag{6}$$

the factor space $\mathfrak{H}_1 = \mathcal{L}_0^\perp/\mathcal{L}_0$ is a J-space and, as was shown, $A\mathcal{L}_0^\perp \subset \mathcal{L}_0$.

Let $x \in \Pi_k$. By virtue of (5), x has the decomposition

$$x = y + z + t, \tag{7}$$

where $y \in \mathcal{F}_0,\ z \in \mathcal{L}_0,\ t \in \mathcal{E}_0^\perp$.

$$Ax = Ay + Az + At. \tag{8}$$

Since $\mathcal{L}_0^\perp = \mathcal{L}_0 \oplus \mathcal{E}_0^\perp$ and $A\mathcal{L}_0^\perp \subset \mathcal{L}_0$, we have $At \in \mathcal{L}_0$ and $Az \in \mathcal{L}_0$. Put $\mathcal{Q}_0 = A\mathcal{F}_0$; and let $\mathcal{N}_0$ be the linear span of the subspaces $\mathcal{L}_0$ and $\mathcal{Q}_0$. Then from (6) and (7), taking into account the inclusions written above, we obtain

$$A\Pi_k \subset \mathcal{N}_0. \tag{9}$$

Moreover $\dim \mathcal{F}_0 = \dim \mathcal{L}_0$, and therefore $\dim \mathcal{F}_0 \leq k$. Hence, $\dim \mathcal{N}_0 \leq 2k$, and Theorem 2 is proved.

Corollary 1. *A J-self-adjoint operator A from Ω in a Pontryagin space is nilpotent and $A^{2k_1+1} = 0$, where k_1 is the dimension of the maximal J-neutral invariant subspace of the operator A, so that $k_1 \leq k$.*

M. A. Naimark in the works ([6],[7]) obtained a description of symmetric commutative algebras R in Π_k. From Corollary 1 and ([6],[7]) we obtain

Corollary 2. *If the algebra R is not semisimple, then the class $\Omega \cap R$ belongs to the radical of the algebra R.*

We now indicate a class of nilpotent operators $\mathfrak{D}(0)$ in $\mathfrak{H}$ for which there exists no J-metric with respect to which the operators of this class are J-self-adjoint.

Theorem 3. *Let $A \neq 0$ be a nilpotent operator in $\mathfrak{H}$. If the imaginary part of the operator A is sign-definite, then there exists no J-metric with respect to which the operator A is J-self-adjoint.*

Proof. Without loss of generality one may assume that

$$\operatorname{Im} A = (A - A^*)/2i \geq 0,$$

since one may pass from the operator A to the operator $-A$. Suppose that there exists a J-metric such that the operator A is J-self-adjoint. The Cayley transform of the operator A is a J-unitary operator U in $\mathfrak{H}$. On the other hand, from the condition $\operatorname{Im} A \geq 0$ we obtain that ([8])

$$\|U\| \leq 1, \tag{10}$$

and then for the operator U there exists (^{9}) an invariant maximal dual pair of subspaces $\{\mathcal{L}, \mathcal{F}\}$ and

$$\mathfrak{H} = \mathcal{L} \oplus \mathcal{F}. \tag{11}$$

The pair $\{\mathcal{L}, \mathcal{F}\}$ will be invariant also with respect to the operator A, and therefore, with respect to the definite scalar product $[x, y]_1$, introduced as in Theorem 1, the operator A becomes self-adjoint. Thus, $A = 0$, which contradicts the condition of the theorem.

Corollary. *If a nilpotent operator A is dissipative, then there exists no indefinite scalar product with respect to which the operator is J-self-adjoint.*

Let us give an example. The integral operator N in $L_2(0,1)$, defined by the equality

$$(Nf)(t) = 2i \int_0^t f(s)\, ds,$$

is dissipative, and therefore there is no operator $J = P_+ - P_-$, $P_+ \oplus P_- = I$, such that $(JN)^* = JN$.

In conclusion I express my sincere gratitude to M. A. Naimark and I. S. Iokhvidov for their attention to this work.

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Received
4 III 1968

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